Tackling Multiplicity of Equilibria with Gröbner Bases

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Multiplicity of equilibria is a prevalent problem in many economic models. Often equilibria are characterized as solutions to a system of polynomial equations. This paper gives an introduction to the application of Gröbner bases for finding all solutions of a polynomial system. The Shape Lemma, a key result from algebraic geometry, states under mild assumptions that a given equilibrium system has the same solution set as a much simpler triangular system. Essentially, the computation of all solutions then reduces to finding all roots of a single polynomial in a single unknown. The software package SINGULAR computes the equivalent simple system. If all coefficients in the original equilibrium equations are rational numbers or parameters, then the Gröbner basis computations of SINGULAR are exact. Thus, Gröbner basis methods cannot only be used for numerical approximations of equilibria, but in fact may allow the proof of theoretical results for the underlying economic model. Three economic applications illustrate that without much prior knowledge of algebraic geometry, Gröbner basis methods can be easily applied to gain interesting insights into many modern economic models.

1. Introduction

Multiplicity of equilibria is a prevalent problem in economics, both in equilibrium models with strategic interactions and in competitive models. Although this problem has long been acknowledged in the theoretical literature, it has in the past often been ignored in applied work. However, recently there appears now also to be a growing interest in equilibrium multiplicity in active areas of modern applied economic analysis. For example, Bodenstein (2008) points out that multiple steady states arise for reasonable parameter values in a standard model of the international business cycle literature. Similarly, Besanko et al. (2010) show that multiple Markov-perfect equilibria can easily arise in a stochastic game model of industry dynamics. This model is an example of a large class of models that has become very popular in industrial organization and marketing, and in many other applications we may often suspect that there could be multiple equilibria. However, standard numerical methods only search for a single equilibrium. There is clearly a need in economics for methods that can find all equilibria for applied models.

In many economic models equilibria can be described as solutions of polynomial equations (that perhaps also must satisfy some additional inequalities). Recent advances in computational algebraic geometry have led to several powerful methods and their easy-to-use computer implementations that find all solutions to polynomial systems. Two different solution approaches stand out: all-solution homotopy methods and Gröbner basis methods. For reasons that we describe below, we focus on Gröbner basis methods in this paper. We provide a fairly nontechnical introduction to these methods and to the computer algebra system SINGULAR, which as of the writing of this paper is considered the best freely available software (www.singular.uni-kl.de) for computing Gröbner bases. Three economic applications illustrate that without much prior knowledge of algebraic geometry, Gröbner basis methods can be applied to gain interesting insights into many modern economic models.

The basic idea of the Gröbner basis methods for solving polynomial systems of equations is as follows. The Shape Lemma, a key result from algebraic geometry, states under mild assumptions that a given equilibrium system has the same solution set as a much simpler triangular system. Essentially, the computation of all solutions reduces to finding all roots of a single polynomial in a single unknown. The software package SINGULAR computes the equivalent simple system. If all coefficients in the original equilibrium equations are rational numbers or parameters then the computations of SINGULAR are exact. Thus, Gröbner bases cannot only be used for a numerical approximation of equilibria, but in fact may allow the proof of theoretical results for the underlying economic model.

Homotopy continuation methods provide an alternative method for finding all solutions. The basic idea is to start at a generic polynomial system $g(x)$ whose number of roots...
is at least as large as the maximal number of solutions to \( f(x) = 0 \) and whose roots are all known. Then one needs to trace out all paths (in complex space) of the homotopy \( H(x, t) = tg(x) + (1 - t)f(x) \) starting at each solution for \( t = 0 \). All solutions to \( f(x) = 0 \) can be found in this manner, see Sturmfels (2002) and Sommese and Wampler (2005).

The software package PHCpack\(^1\) provides a fast and robust implementation of an all-solution homotopy method among its many features. The solver can be used as a black box, entering the system of polynomial equations in a file. Homotopy methods can usually solve larger systems, with more unknowns and polynomials of higher degrees, than Gröbner basis methods. However, Gröbner basis methods have the following important advantages.

1. Homotopy methods are purely numerical methods. Due to rounding errors, it is sometimes difficult to determine whether a homotopy path has indeed converged or whether the final numerical solution is real or complex. On the contrary, Gröbner basis methods offer the possibility of exact calculations without any rounding errors. Therefore, such a method may allow us to prove the existence of a unique equilibrium.

2. We can calculate Gröbner bases for parameterized polynomials. Thus, we can establish bounds on the number of equilibria for entire classes of economic models.

3. Parameterized Gröbner bases enable us to search for specific parameter values for which there are multiple equilibria or to prove that equilibria are unique for all parameter values in a given set.

We illustrate these points with three economic examples. Although these are small examples that are chosen to illustrate the advantages of Gröbner bases, they also provide some interesting economic insights. Hopefully, they serve as a motivation for the reader to apply the methods presented in this paper to other and larger models.

We first compute Arrow-Debreu equilibria for economies with two types of agents having heterogeneous CES utility and heterogeneous endowments. We show how the number of equilibria changes in response to changes in the endowments.

Secondly, we consider a game under incomplete information with cheap talk (the arms race game of Baliga and Sjöström 2004) and show how multiplicity of perfect Bayesian equilibria in such a game can be addressed with Gröbner bases. Agents’ types are i.i.d. with a cumulative distribution function \( F \). Under the assumption that \( F \) is polynomial, the solutions of a system of polynomial equations constitute cutoff values in type space, which in turn determine the cheap-talk message that agents send in equilibrium. We compute the Gröbner basis for the polynomials appearing in the equations and then compute all solutions. We show how the two equilibria in the game change as some key parameter in the model changes.

Finally, we give an example of multiplicity of steady states in a general equilibrium model with overlapping generations and individuals that live for more than two periods. For a class of models, we show that there can never be more than three steady states and give examples of models where this bound is actually attained. Models of overlapping generations are routinely used in applied policy analysis (see, e.g., Auerbach and Kotlikoff 1987). The multiplicity of steady states in these models potentially casts doubts on the validity of this analysis. Our analysis illustrates that examples of multiplicity can easily be constructed in standard models, but also that it is true that steady states are unique for a large range of parameter values.

There is a growing literature on the computation of all equilibria in normal form games; see Datta (2010) for an excellent recent survey and also Sturmfels (2002) and Herings and Peeters (2005). In this paper, we do not address this problem because we consider it somewhat less important in applied economic modeling.

This paper is organized as follows. In §2 we give a simple nontechnical introduction to Gröbner bases. Section 3 describes how to use SINGULAR to compute Gröbner bases and all solutions to polynomial equations. In §§4–6 we provide examples of interesting economic applications.

### 2. Some Background on Polynomials and Gröbner Bases

In this section we summarize some basic definitions and concepts from the field of algebraic geometry that are fundamental to our analysis in this paper. We refer the interested reader to the textbooks by Cox et al. (1997, 1998). The treatment in this section is deliberately simple.

#### 2.1. Polynomials

For the description of a polynomial \( f \) in the \( n \) variables \( x_1, x_2, \ldots, x_n \) we first need to define monomials. A monomial in \( x_1, x_2, \ldots, x_n \) is a product \( x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) where all exponents \( \alpha_i \), \( i = 1, 2, \ldots, n \), are nonnegative integers. It will be convenient to write a monomial as \( x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) with \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n \), the set of non-negative integer vectors of dimension \( n \). A polynomial is a linear combination of finitely many monomials with coefficients, in a field \( \mathbb{K} \). In this paper we do not need to consider arbitrary fields of coefficients, but instead we can focus on three commonly used fields. These are the field of rational numbers \( \mathbb{Q} \), the field of real numbers \( \mathbb{R} \), and the field of complex numbers \( \mathbb{C} \). Polynomials over the field of rational numbers are computationally convenient because modern computer algebra systems, such as SINGULAR, perform exact computations over the field \( \mathbb{Q} \). Economic parameters are typically real numbers, and thus force us to consider \( \mathbb{R}[x] \).

We can write a polynomial \( f \) as \( f(x) = \sum_{\alpha \in S} a_\alpha x^\alpha \), with \( a_\alpha \in \mathbb{K} \) and \( S \subset \mathbb{Z}_+^n \) finite. We denote the collection of all polynomials in the variables \( x_1, x_2, \ldots, x_n \) with coefficients in the field \( \mathbb{K} \) by \( \mathbb{K}[x_1, \ldots, x_n] \), or, when the dimension is clear from the context, by \( \mathbb{K}[x] \). The set \( \mathbb{K}[x] \) is called
“a polynomial ring” (it satisfies the properties of a so-called commutative ring).

We are interested in the set of real solutions to a system of polynomial equations, that is, given $f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n]$, we want to find all elements in (the hopefully finite) set $\{ x \in \mathbb{R}^n : f_1(x) = \cdots = f_k(x) = 0 \}$. The study of solution sets of polynomial equations requires the variables $x$ to range over an algebraically closed field. Unlike the fields $\mathbb{Q}$ and $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ is algebraically closed. Therefore, we start off by examining the set of all complex solutions,

$$ V = \{ x \in \mathbb{C}^n : f_1(x) = \cdots = f_k(x) = 0 \}. $$

This solution set is called the complex variety defined by $f_1, \ldots, f_k$. The key observation in finding all solutions is that we can multiply each of the polynomials $f_i$ by another nonzero polynomial and add any polynomials $f_i$ and $f_j$ without eliminating any of the original solutions and without introducing additional solutions. It turns out that Gaussian elimination in linear algebra has a close analogue for polynomial equations. In order to make this intuition more formal, we need an additional definition. For given polynomials $f_1, \ldots, f_k$, the set

$$ I = \left\{ \sum_{i=1}^k h_i f_i : h_i \in \mathbb{K}[x] \right\} = \langle f_1, \ldots, f_k \rangle $$

is called the ideal generated by $f_1, \ldots, f_k$. The ideal $\langle f_1, \ldots, f_k \rangle$ is the set of all linear combinations of the polynomials $f_1, \ldots, f_k$, where the “coefficients” in each linear combination are themselves polynomials in the polynomial ring $\mathbb{K}[x]$. Two aspects about ideals are crucial for our analysis. First note that

$$ \{ x \in \mathbb{C}^n : f_1(x) = \cdots = f_k(x) = 0 \} = \{ x \in \mathbb{C}^n : g(x) = 0 \text{ for all } g \in \langle f_1, \ldots, f_k \rangle \}. $$

In other words, the set of solutions to a polynomial system of equations is identical to the set of solutions to all (infinitely many!) polynomials in the ideal generated by the system. Therefore, we can call the solution set $V$ the complex variety of the ideal $\langle f_1, \ldots, f_k \rangle$. Secondly, note that we can find other polynomials, $g_1, \ldots, g_r$, such that $\langle g_1, \ldots, g_r \rangle = \langle f_1, \ldots, f_k \rangle$ and

$$ \{ x \in \mathbb{C}^n : f_1(x) = \cdots = f_k(x) = 0 \} = \{ x \in \mathbb{C}^n : g_1(x) = \cdots = g_r(x) = 0 \}. $$

The sets of polynomials $g_1, \ldots, g_r$ and $f_1, \ldots, f_k$ are called bases of the ideal $\langle f_1, \ldots, f_k \rangle$. The idea is then to find an alternative basis for the ideal generated by $f_1, \ldots, f_k$ that is easy to solve. Consider the example of two polynomials $f_1$ and $f_2$ in the two unknowns $x$ and $y$. $f_1 = 2x^2 + 3y^2 - 11$ and $f_2 = x^2 - y^2 - 3$. What can we say about the set of common roots of these two polynomials? Note that for any ideal $I$, simply by definition, if $f_1, \ldots, f_k \in I$, then $\langle f_1, \ldots, f_k \rangle \subset I$. Thus, showing $f_1, f_2 \in \langle x^2 - 4, y^2 - 1 \rangle$ and conversely $x^2 - 4, y^2 - 1 \in \langle f_1, f_2 \rangle$ proves that $\langle f_1, f_2 \rangle = \langle x^2 - 4, y^2 - 1 \rangle$. Therefore, $V(f_1, f_2)$ consists of the four points $(2, 1), (-2, 1), (2, -1), (-2, -1)$.

Obviously, the example is rather simple and we could have solved the problem without any knowledge of the term “ideal.” However, the solution approach of transforming a given system of polynomial equations into a simpler system with an identical solution set works much more generally. Under some mild conditions it is always possible to find an alternative basis for a polynomial system that can be solved easily. Such a good basis is the so-called reduced Gröbner basis under lexicographic monomial order. In the remainder of this section, we discuss its properties.

### 2.2. The Exact Computation of All Solutions

For the remainder of this paper we restrict ourselves to square systems of polynomial equations, that is, $k = r = n$. In a slight abuse of notation, we call an ideal regular if its complex variety has finitely many complex solutions that are locally unique in the sense that the Jacobian has full rank at all solutions. That is, we call $I$ regular at $x \in \mathbb{C}^n$ if for $f = (f_1, f_2, \ldots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ it holds that $f(x) = 0 \Rightarrow D_x f(x)$ has full rank $n$. Also, we call $I$ regular if it is regular for all $x \in \mathbb{C}^n$. With this definition we can state the key result for our analysis in this paper (see Becker et al. 1994 for a proof).

**Lemma 1 (Shape Lemma).** Let $I$ be a regular ideal in $\mathbb{Q}[x_1, \ldots, x_n]$ with all $d$ complex roots of $I$ having distinct $x_n$ coordinates. Then there exists a basis with the shape

$$ \mathcal{B} = \{ x_1 - q_1(x_n), x_2 - q_2(x_n), \ldots, x_n - q_n(x_n), r(x_n) \}, $$

where $r$ is a polynomial of degree $d$ and the $q_i are polynomials with a degree of at most $d - 1$.

The special type of basis described by the Shape Lemma is an example of the aforementioned reduced Gröbner basis of $I$ in the lexicographic monomial order. If the lemma holds, finding all solutions to a polynomial system of equations reduces to finding all solutions of a single equation, a task for which there exist efficient numerical methods. The question is therefore whether the assumptions of the lemma, namely (i) the ideal being regular and (ii) all roots having distinct $x_n$ coordinates, are easy to verify and likely to hold in economic models.

The first condition is typically difficult to verify because it means that there are only finitely many complex solutions that are all regular. In many economic applications we are only interested in real solutions and we can show that there are finitely many of them, or we are only interested in positive real solutions and can prove finiteness of these. In such cases we can add an equation in order to eliminate the unwanted real and complex solutions that may be the reason for a violation of the first assumption. A simple
approach to ensure that all solutions are regular is to add the following additional polynomial equation to the original system:

\[ 1 - t \det [D_x f(x)] = 0. \]

There cannot be a solution in \( t \) and \( x \) that is not locally unique.

Condition (ii) holds for a wide variety of problems. In case the condition does not hold, we can always add an additional equation:

\[ y - \sum_{i=1}^{n} a_i x_i = 0. \]

For generic \( (a_1, \ldots, a_n) \), all solutions to \( f(x) = 0 \) and \( y - \sum_{i=1}^{n} a_i x_i = 0 \) will have distinct \( y \)-coordinates and the Shape Lemma holds for the larger system with \( y \) as the last coordinate.

### 2.3. Bounding the Number of Zeros

The Shape Lemma implies that the number of real solutions to \( f_1(x) = \cdots = f_n(x) = 0 \) is equal to the number of real roots of the last polynomial \( r(x) \). The Fundamental Theorem of Algebra, see Sturmfels (2002), states that any univariate polynomial, \( \sum_{i=0}^{d} a_i z^i \), with \( a_i \in \mathbb{R} \) for all \( i \), has \( d \) complex roots (counting multiplicities of zeros). There are bounds available for the number of real zeros. Define the number of sign changes of \( r \) to be the number of elements of \( \{a_i \neq 0, i = 0, \ldots, d - 1: \text{sign}(a_i) = -\text{sign}(a_{i+1}) \} \). The classical Descartes’ Rule of Signs, see Sturmfels (2002), states that the number of real positive zeros of \( r \) does not exceed the number of sign changes.

Moreover, if all \( a_i \in \mathbb{Q} \), Sturm’s theorem provides an algorithm to determine the exact number of real roots of any univariate polynomial in a particular interval (see again Sturmfels 2002). This fact implies that it is possible to state the exact number of equilibria for a given economic model.

### 2.4. A Parametric Shape Lemma

Gröbner bases are particularly useful for theoretical work because we can compute a Gröbner basis for polynomials whose coefficients are parameters. This implies that we can prove statements on the number of equilibria for classes of economic models, see Proposition 1. The following lemma generalizes the Shape Lemma and enables us to represent equilibria of parameterized classes of economic models in the shape form. For the statement of this lemma, we restrict the field \( \mathbb{K} \) to be either the field \( \mathbb{Q} \) of rational or the field \( \mathbb{R} \) of real numbers and extend the definition of the polynomial ring \( \mathbb{K}[x] \) with coefficients in the field \( \mathbb{K} \) to allow for coefficients that are polynomials in parameters \( e_1, \ldots, e_m \).

We denote this ring by \( \mathbb{K}[e; x] \).

**Lemma 2 (Parameterized Shape Lemma).** Let \( E \subset \mathbb{R}^m \) be an open set of parameters, \( (x_1, \ldots, x_n) \in \mathbb{C}^n \) a set of variables, and let \( f_1, \ldots, f_n \in \mathbb{K}[e_1, \ldots, e_m; x_1, \ldots, x_n] \) with \( \mathbb{K} \in \{ \mathbb{Q}, \mathbb{R} \} \). Assume that for each \( \bar{e} = (\bar{e}_1, \ldots, \bar{e}_n) \in E \), the ideal \( I(\bar{e}) = (f_1(\bar{e}; \cdot), \ldots, f_n(\bar{e}; \cdot)) \) is regular and all complex solutions have distinct \( x_i \) coordinates. Then there exist \( r, v_1, \ldots, v_{m-1} \in \mathbb{K}[e; y] \) and \( p_1, \ldots, p_{m-1} \in \mathbb{K}[e] \), not identically equal to zero, such that for all \( \bar{e} \in E \) with \( p_1(\bar{e}) \neq 0 \), for all \( l \) and with \( r(\bar{e}, \cdot) \) not identically equal to zero, the following holds:

\[
\{ x \in \mathbb{C}^n : f_1(\bar{e}, x) = \cdots = f_n(\bar{e}, x) = 0 \}
\]

\[
= \{ x \in \mathbb{C}^n : r_1(\bar{e}) x_1 = v_1(\bar{e}, y), \ldots, p_{m-1}(\bar{e}) x_{m-1} = v_{m-1}(\bar{e}; y), \bar{r}(\bar{e}; x_{m}) = 0 \}. \]

We illustrate the lemma with several examples below. The lemma enables us to treat the coefficients of a polynomial system as parameters and obtain a Gröbner basis where the coefficients of the polynomials in the basis are polynomials in the parameters.

To apply both the standard Shape Lemma and the Parameterized Shape Lemma for the computation of all solutions we need an algorithm that allows us to derive the simple bases in shape form. Buchberger’s algorithm does exactly that. A detailed discussion of this algorithm is beyond the scope of the present paper; we refer the interested reader to Cox et al. (1997, 1998) and Greuel and Pfister (2007) as well as Greuel et al. (2005).

### 2.5. Parametric Gröbner Bases

A difficulty with parametric Gröbner bases is that they may not specialize for all possible parameter values. That is, after substituting specific values for the parameters, the resulting basis may, in fact, not be a Gröbner basis for the original set of polynomials. For a correct specialization at given parameters \( \bar{e} \) it suffices to assume that \( p_1(\bar{e}) \neq 0 \), for all \( l \), that the polynomial \( r \) is not identically equal to zero at \( \bar{e} \), and that \( I \) is regular at all \( \bar{e} \).

The following argument shows why these conditions are sufficient. For \( e \in E \), let \( f_1 = (f_1(e; \cdot), \ldots, f_n(e; \cdot)) \) be a set of polynomials with parameters \( e \) and unknowns \( x_1, \ldots, x_n \). Let \( g_e = (g_1(e; \cdot), \ldots, g_n(e; \cdot)) \) denote the output of Buchberger’s algorithm. The parametric Shape Lemma requires that for all \( e \in E \) the original system \( f \) is regular. Under this assumption, \( g_e \) is the correct reduced Gröbner basis and has the shape form for “generic” \( \bar{e} \), that is, for all \( \bar{e} \in E \) outside a closed set of Lebesgue measure zero (see, e.g., Cox et al. 1997 for this result). We can state a slightly stronger result by explicitly characterizing the set of parameters for which the output of Buchberger’s algorithm is not the correct Gröbner basis.

The implicit function theorem implies that regularity of \( f \) guarantees for all \( \bar{e} \) that for any \( \bar{x} \) among the finitely many solutions to \( f(\bar{e}, \bar{x}) = 0 \), there is an open neighborhood around \( \bar{e} \) such that for all sequences of parameters \( (e^i) \) in this neighborhood that converge to \( \bar{e} \), there exist \( (x^i) \) such that for all \( i \), \( f(e^i, x^i) = 0 \) and \( x \to \bar{x} \). Therefore, if \( f(\bar{e}, \bar{x}) = 0 \), we must also have \( g(\bar{e}, \bar{x}) = 0 \). This is true...
because we can find sequences of parameters that converge to \( \hat{e} \) and for which \( g \) is the correct Gröbner basis at all points along the sequence.

However, in principle it can be the case that \( g(\hat{e}, x) = 0 \) has solutions that are not solutions to the original system. One simple way to rule this out is to require that \( g \) itself is regular at \( \hat{e} \). Rather than assuming this, one can actually verify this. A singularity can only occur if either a derivative with respect to \( x_1, \ldots, x_{n-1} \) is equal to zero, which means that some \( p_i(\hat{e}) = 0, i = 1, \ldots, n-1 \), or that \( r(\hat{e}, x) = 0 \) and \( \partial r / \partial x = 0 \). If \( r(\hat{e}, \cdot) \) is not identically equal to zero, the latter implies that there is a multiple solution that is ruled out by regularity of \( f(\hat{e}, \cdot) \).

The assumption of the lemma that \( I \) is regular for all parameters in \( E \) is obviously very strong. In economic applications, one can typically show regularity for “almost all” parameter values, i.e., for parameters outside of a closed set of measure zero. Therefore, naturally the question arises what happens at some \( \hat{e} \) where \( I \) is not regular, but where in any open neighborhood of \( \hat{e} \) there is a regular \( I \). As the following example shows, one cannot find the solution set \( V(I) \) from knowing \( V(G) \), even if \( G \) is regular. Suppose as a trivial example \( I = \langle e(x+y), xy+e \rangle \subset \mathbb{Q}[e; x, y] \). A valid Gröbner basis for almost all parameters \( e \) is given by \( G = \langle y^2 - e, x + z \rangle \). Clearly, for \( e = 0 \), this is not the correct Gröbner basis. This fact does not contradict the lemma because \( I \) is not regular at 0.

Instead of assuming the regularity of the ideal \( I \), we could compute the set of parameter values for which the Gröbner basis may not specialize correctly. Cox et al. (1997, pp. 283–284) describe an algorithm for this purpose. Another approach to address this problem is to only assume that there are finitely many complex solutions, but not to assume that these solutions are all regular. In this case, the resulting ideal might not be a so-called radical ideal; see Cox et al. (1997). However, there exist algorithms (implemented in \textsc{singular}) that can compute the radical of the ideal, i.e., that compute a system of polynomial equations that has the same (finitely many) complex zeros but that has the property that at all solutions the Jacobian has full rank. We give an example of this in §3.3 below.

Having a shape representation of the polynomial system in parameters has two advantages. Often one can use Descartes’ method to derive bounds on the number of equilibria uniform over all (or almost all) parameters. Secondly, one can search for parameters for which the system has a critical point. At these points the number of solutions typically changes (more precisely, if the number of solutions changes, it must be at a critical point).

### 2.6. Detecting Multiple Solutions

In many situations it is not enough to verify that for a given \( \hat{e} \) the system of equations \( f(\hat{e}, x) = 0 \) has a unique real solution, but in fact we want to prove that for almost all \( e \) in some (convex) set of parameters \( E \), real solutions are unique. Formally, we want to rule out that there is some open set \( M \subset E \) such that for all \( e \in M \), the system \( f(e, x) = 0 \) has multiple real solutions. We refer to this as robust (real) uniqueness.

For a given parametric Gröbner basis with univariate representation \( r(e; x_e) \), we say that a critical point (of the Gröbner basis) occurs at \( \hat{e}, \hat{x}_e \) with \( \hat{x}_e \in \mathbb{R} \) if and only if

\[
\frac{\partial r(\hat{e}, \hat{x})}{\partial x_e} = 0.
\]

(1)

Suppose for parameters \( \hat{e} \in E \) the system of equations has a unique real solution. Robust multiplicity of real solutions can now occur in \( E \) only if the above system of equations has a real solution in \( E \). At such points the parametric Shape Lemma no longer holds and does not allow us to draw any conclusions about the correct Gröbner basis. However, for an open set of parameters to exist for which the original system has multiple real solutions, there must be an open set for which the Gröbner basis has multiple solutions. This, in turn, can only occur if the Gröbner system has a critical point and leads us to the problem of detecting such points in the Gröbner basis.

For a single free parameter \( e \in \mathbb{R} \), system (1) consists of two equations in two unknowns, and the equations typically have finitely many solutions that can be found by computing the Gröbner basis of this new system. In the applications below, we focus on this simple case. For several parameters, the solution set is infinite (i.e., positive dimensional). Aubry et al. (2002) develop an algorithm to find representative solutions of the positive-dimensional system.

### 3. Gröbner Basis Computation with \textsc{singular}

As of the writing of this paper, the computer algebra system \textsc{singular} is considered to be among the leading, if not the best, freely available software package for Gröbner basis computations. Decker and Lossen (2006), Greuel and Pfister (2007), and Greuel et al. (2005) provide detailed descriptions of this software. \textsc{singular} has many capabilities that we do not need for our objective of finding all economic equilibria. Therefore, here we provide only information on the software that is needed for the computation of Gröbner bases and all solutions to square systems of polynomial equations.

Consider the following system of three equations in the three unknowns \( x, y, z \):

\[
x - yz^3 - 2z^4 + 1 = -x + yz - 3z + 4 = x + yz^2 = 0.
\]

The polynomials on the left-hand side of the three equations define a polynomial ideal. As a first step we compute a Gröbner basis for this ideal. We enter the following commands in \textsc{singular}:

```
ring R=0,(x,y,z),lp;
ideal I=(x-y*z**3-2*z**4+1, -x+y*z-3*z+4, x+y*z**2);
ideal G=groebner(I);
```
In \textsc{Singular}, we first have to declare a base ring that we call $R$ in this example. A zero in the ring declaration indicates that we consider polynomials over the rational numbers. We highly recommend this declaration because only then is the computation exact. We denote the unknowns by $x$, $y$, $z$ and use “lp” to instruct \textsc{Singular} that we use the lexicographic monomial order for the variables; see Cox et al. (1997) for background on monomial orderings. This first command line should only be altered to change the number (or names) of the variables or to introduce parameters. Next, an ideal, here called $I$, is defined via a list of the polynomials that form a basis of the ideal. Note that \textsc{Singular} requires the signs $*$ and $**$ to indicate the multiplication and power operation, respectively. The command \texttt{groebner(I)} first chooses an “optimal” algorithm under several available and then computes a Gröbner basis. To view the output, we type “$G;$” at the \textsc{Singular} command line and obtain the following output:

$G[1] = 2z11+3z9-5z8+5z3-4z2-1$

Contrary to the input format, the output of \textsc{Singular} does not use the multiplication and power signs. For example, $2z11$ is to be read as $2 \cdot z^{11}$. In this example, the Shape Lemma holds. There are 11 solutions because $G[1]$ is a polynomial of degree 11 in the last variable $z$. Note that by Descartes’ bound, at most three of them can be real and positive. To solve numerically for all complex solutions, we first need to load a library in \textsc{Singular}.

\begin{verbatim}
>LIB "solve.lib"
>
>solve(G);

Only one of the 11 solutions is real, the other 10 are complex. \textsc{Singular} prints them all. The unique real solution is $(1/2, -1/2, 1)$.

3.1. Parameters

The following variation of the previous example illustrates how to introduce parameters. Let the coefficient of the monomial $x$ in the last equation be a free parameter, i.e., the last equation becomes $ex + yz^9 = 0$. In \textsc{Singular} we now need to declare the parameter as part of the ring in the initial command line. The declaration “$R=\mathbb{Q}(e)$” states that all polynomial coefficients contain only elements of $\mathbb{Q}$ and the parameter $e$.

\begin{verbatim}
ring R=\mathbb{Q}(e),(x,y,z),lp;
ideal I=(x+y*z**3-2*z**3+1, -x+y*z-3*z+4, 
   e*x+y**z**9);
ideal G=groebner(I);
G;
G[1]=2z11+3z9-5z8+(5e)*z3+(-4e)*z2+(-e)
G[2]=(-e2-e)*y+(-8e-10)*z10+(-10e-15)*z8 
   +(20e+25)*z7+(5e)*z6+(5e)*z5+(5e)*z4+(-5e) 
   *z3+(-20e-20e)*z2+(16e2+15e)*z+(3e2+3e)
G[3]=(-e-1)*x+2z9+5z7-5z6+5z5-5z4+5z3-5z2-1.
\end{verbatim}

\textsc{Singular} produces a Gröbner basis for the ideal of parameterized polynomials. Observe that the univariate representation $G[1]$ is a polynomial of degree 11 for any value of $e$. Figure 1 shows the real roots of the univariate representation for $e \in [-3, 1]$. For positive values of $e$, $G[1]$ has the unique solution $z = 1$. For nonpositive values of $e$ there are multiple solutions. Before we decide on the number of real solutions for specific values of $e$, we must not forget that for fixed values of the parameter the parameterized Gröbner basis may not specialize to the correct basis. Here this difficulty becomes obvious. Observe that the leading term of $G[2]$ is $(e-e-1)y$ and so for $e \in (-1, 0)$ the variable $y$ no longer appears. The same is true for the variable $x$ in $G[3]$ for $e = -1$. Figures 2 and 3 show the real solutions for $G[2]$ and $G[3]$, respectively, for $e \in [-3, 1]$.

As $e \rightarrow -1$ the values of $y$ and $x$ grow unbounded in two of the three solutions. Only in one solution do their values remain bounded. For $e = -1$, both variables no longer appear in the Gröbner basis. As $e \searrow 0$, the values of $y$ and $x$ remain bounded in all three solutions.

Instead of using the parameterized basis, we need to resolve the original system for $e = 0$ and $e = -1$. For $e = 0$ the resulting Gröbner basis is $\{2z^3+3z-5, y, x+3z-4\}$. There is a unique real solution, $(1, 0, 1)$. This indicates that

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Real solutions for $z$ depending on the parameter $e$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Solutions for $G[2]=0$ for real values of $z$.}
\end{figure}
as $e \neq 0$ two of the three solutions do not converge to a solution even though all three solutions remain finite. Only the solution with $z = 1$ converges to a solution of the original system at $e = 0$. For $e = -1$ the Gröbner basis is as follows:

$$G[1] = 2z^9 + 5z^7 - 5z^6 + 5z^5 - 5z^4 + 5z^3 - 5z^2 - 1$$
$$G[2] = 33y + 320z^8 + 102z^7 + 790z^6 - 765z^5 + 740z^4 - 715z^3 + 690z^2 - 665z - 94$$
$$G[3] = 33x + 102z^8 - 102z^7 + 35z^6 - 60z^5 + 85z^4 - 110z^3 + 135z^2 + 5z + 28.$$

There is a unique real solution, $(-3.37023, -4.63605, 0.965189)$.

### 3.2. Critical Points

If along a path of parameters the number of real solutions changes, then there must be a critical point—that is, system (1) with the parameters and the last variable $x_n$ as its unknowns must have a solution. For a single parameter, this system consists of two equations in two unknowns. For our example, the appropriate Singular code is then as follows:

```plaintext
ring R=0,(e,z),lp;
ideal I=(2*z**11+3*z**9-5*z**8+5*e*z**3-4*e*z**2-e,11*z**10+9*z**8-8*z**7+3*5*e*z**2-2*4*e*z);
ideal G=grobner(I);
solve (G);
```

Observe that we now must declare the parameter $e$ as one of the two variables. This system has two real solutions, $(e, z) = (0, 0)$ and $(e, z) = (-9/7, 1)$. For $e = 1$ we have seen that there is a unique solution, and so there must be a unique solution for all $e > 0$. In order to have two solutions for any $e > 0$, there must be a critical point between $e$ and $e = 1$, which is not the case. For all positive parameters the system has a unique solution. At $e = -9/7 \approx -1.28571$ there is a critical point, but the number of solutions is three both for smaller and larger values of $e$. At that point, there is a multiple solution.

### 3.3. Failure of Shape Lemma

The Shape Lemma rests on the two assumptions that the ideal $I$ is regular and that all solutions have distinct values for the last variable. Here we illustrate with two simple examples what can happen when the assumptions are not satisfied.

Consider the system of equations $x^2 - y = y - 4 = 0$. This system has the two solutions $(-2, 4)$ and $(2, 4)$. If $y$ is the last coordinate, then both solutions have the same value for the last coordinate. No polynomial that is linear in $x$ of the form $x - q(y)$ can yield two different solutions for $x$ for the same value of $y$. It is easy to see (without any computation) that the Gröbner basis is as follows:

$$G[1] = y - 4$$

The Shape Lemma fails, $G[2]$ is not linear in $x$. However, observe that if we reorder the variables, then the Shape Lemma holds because now $x$ is the last coordinate and both solutions have different values for $x$.

Consider the system of equations $x^2(x - y) = y^2(x + y - 1) = 0$. Clearly, for $x = y = 0$ the solutions are not locally unique in our sense. The Shape Lemma fails and the Gröbner basis is as follows:

$$G[1] = 2y^5 - 5y^4 + 4y^3 - y^2$$
$$G[2] = xy^2 + y^3 - y^2$$

However, it is easy to see that the system has finitely many solutions. To solve for them, we need to compute what is called the “radical” of the system, see Cox et al. (1997). After loading the library solve.lib we can also use the command radical and obtain a Gröbner basis with multiple zeros eliminated:

```plaintext
ideal J=grobner(radical(I));
J[1]=2y^3-3y^2+y
```

The Shape Lemma holds once we compute the radical of the ideal. This approach works if and only if all the solutions have a distinct last coordinate and there are finitely many. Our notion of regular ideal requires two elements, namely, finitely many solutions and full rank of the Jacobian at all solutions. The latter condition automatically holds if we first compute the radical of the ideal.

### 4. Multiple Arrow-Debreu Equilibria

Our first example concerns the computation of all equilibria in the standard Arrow-Debreu general equilibrium model; see textbooks such as Mas-Colell et al. (1995). Suppose there are two types of agents and two commodities. Utility functions are

$$u^1(c_1, c_2) = -\frac{64}{2} c_1^{-2} - \frac{1}{2} c_2^{-2},$$
$$u^2(c_1, c_2) = \frac{1}{2} c_1^{-2} - \frac{64}{2} c_2^{-2}. \tag{2}$$
Parameterized individual endowments are \( e^1 = (1 - e, e) \) and \( e^2 = (e, 1 - e) \) for the parameter \( e \in [0, 1] \). We denote the endowment and consumption of agent \( h \) in good \( l \) by \( e_h^l \) and \( c_h^l \), respectively. The price of good \( l \) is \( p_l \). Using the necessary and sufficient first-order conditions for the agents’ utility maximization problems in addition to the market-clearing equations yields the following equilibrium system, where \( \lambda_h \) denotes the Lagrange multiplier for the budget constraint of agent \( h \):

\[
\begin{align*}
64c_{11}^3 - \lambda_1 p_1 &= 0, \\
c_{12}^3 - \lambda_1 p_2 &= 0, \\
p_1(c_{11} - e_{11}) + p_2(c_{12} - e_{12}) &= 0, \\
c_{21}^3 - \lambda_2 p_1 &= 0, \\
64c_{22}^3 - \lambda_2 p_2 &= 0, \\
p_1(c_{21} - e_{21}) + p_2(c_{22} - e_{22}) &= 0, \\
\end{align*}
\]

We transform this equilibrium system into a much simpler system of polynomial equations. By Walras’ law we can normalize prices by setting \( p_1 = 1 \) and eliminate the budget equation of agent 2. This normalization allows us to eliminate the Lagrange multipliers and the first optimality condition of each agent. Next, the market-clearing equation allows us to eliminate the consumption variables of the second agent. Finally, we substitute \( p_2 = q^3 \) and write the remaining equations in terms of the excess demand \( x_1 = c_{11} - e_{11} \) of agent 1 to obtain the following polynomial system:

\[
\begin{align*}
(1 - e + x_1) - 4(e + x_2)q &= 0, \\
4(e - x_1) - (1 - e - x_2)q &= 0, \\
x_1 + x_2q^3 &= 0.
\end{align*}
\]

The corresponding singular code and the resulting Gröbner basis are as follows:

```singular
ring R = (0,e),(x(1),x(2),q),lp;
option(redSB);
ideal I = (-4*(e+x(2))*q+(1-e+x(1)), -(1-e-x(2))*q+4*(e-x(1)), x(1)+x(2)*q^3);
ideal G=groebner(I);
G[1]=-15*e-1)*q^3+4*q^2-4*q+(15*e+1)
G[2]=-225*q*e-15)*x(2)+(60*e+4)*q^2-16*q
+(-225*e^2-30*e+15)
G[3]=15*x(1)+4*q^2+(-15*e-1).
```

The univariate representation has three roots,

\[
\begin{align*}
1, \\
3 - 15e - \sqrt{5(1 - 42e - 135e^2)} \\
2(1 + 15e) \\
3 - 15e + \sqrt{5(1 - 42e - 135e^2)} \\
2(1 + 15e)
\end{align*}
\]

Figure 4 displays the real roots of \( G[1] \) for small values of \( e \). For \( e \leq 1/45 \) all three roots are real and positive. The corresponding values for the excess demand variables, \( x_1, x_2 \), lead to positive consumption values for both agents. Thus, there are three Arrow-Debreu equilibria. For \( e = 1/45 \) the polynomial \( G[1] \) has \( q = 1 \) as a triple root. For \( e > 1/45 \), two of three roots are complex, and so \( q = 1 \) remains as the unique Arrow-Debreu equilibrium. Note that for positive values of \( e \) all leading terms are never zero, and so the parameterized Gröbner basis specializes correctly.

5. Multiple Perfect Bayesian Equilibria

As a second application of the Gröbner bases methods, we compute equilibria in a game-theoretic model with cheap talk. Specifically, we compute Bayesian Nash equilibria for the arms race game of Baliga and Sjöström (2004), hereafter BS. For this purpose we first summarize the computationally relevant aspects of the game. For many additional details and particularly the interpretation of the game, we refer the reader to the original paper.

5.1. An Arms Race Game with Cheap Talk

Two players simultaneously and independently choose between building a new weapons system (\( B \)) and not building new weapons (\( N \)). If both players choose \( N \), then the payoff to each of them is 0. A player who chooses \( N \) while its opponent chooses \( B \) suffers a loss \( d > 0 \). A player who chooses \( B \) while its opponent chooses \( N \) receives a gain of \( \mu > 0 \). Player \( i \)’s cost of acquiring a new weapons system is \( c_i \geq 0, \ i = 1, 2 \). Player \( i \)’s payoffs are summarized in the following payoff matrix, where player \( i \) chooses a row and its opponent \( j \) a column:

\[
\begin{array}{c|ccc}
B & N \\
\hline
-\mu-c_i & -c_i \\
N & d & 0 \\
\end{array}
\]

The cost \( c_i \) is player \( i \)’s private information. We refer to \( c_i \) as player \( i \)’s type. Each player knows its own type \( c_i \),
but not the other player’s type c_j. The types c_1 and c_2 are
i.i.d. with a continuous cumulative distribution function F.
The function F has compact support [0, c̄] with F(0) = 0, F′(c) > 0 for 0 < c < c̄, and F(c̄) = 1. Also, c̄ < d. All
parameters and functions are common knowledge, with the
exception of the types c_1 and c_2.

For their analysis of the arms race game, BS introduce
a key assumption, the multiplier condition for the distribu-
tion function F. This condition requires that F(c)d \geq c for
all c \in [0, c̄]. BS show that if the multiplier condition is
satisfied, then for any \mu > 0 there is a unique Bayesian-
Nash equilibrium. In this equilibrium all players choose to
build a new weapons system (action B), regardless of their
type. In the language of BS, the only equilibrium outcome
in the game is an “arms race.” This outcome is inefficient
because all types prefer (N, N) to the equilibrium (B, B).

After the analysis of the described equilibrium outcome,
BS introduce cheap talk to their game. The cheap-talk exten-
sion function F is a polynomial function on its support
has compact support 0 < c < c̄. BS show that in equilibrium, if both players
have a type exceeding c_H, then they both send a conciliatory
message in stage one and play N in stage two. In the interpretation
of BS, contrary to the original game without
cheap talk, there is now an equilibrium in which an arms
race is avoided. Also, if c_H \to 0, then the probability of
player i having a type c_i > c_H tends to one, 1 − F(c_H) \to 1.
This is the key result of the paper.

BS point out that there may be other perfect Bayesian
equilibria of the cheap-talk extension of the arms race
game. They mention a second equilibrium with cutoffs
(c_L^∗, c_H^∗, c^H) that also satisfy Equations (4)–(6). This second
equilibrium has the property that (c_L^∗, c_H^∗) \to (0, c^{med})
as \mu \to 0, where c^{med} is the median cost given by
F(c^{med}) = 1/2. Thus, in this second equilibrium an arms
race can be avoided for small \mu only roughly a quarter of
the time. BS make no statements regarding whether there
are other equilibria than the two described ones.

5.2. Polynomial Specification of the Game

Any solution to Equations (4)–(6) that also satisfies the
inequalities (3) yields an equilibrium of the structure
described by BS. Clearly, we cannot hope to solve these
two equations in three unknowns for general functions F.
However, if F is a polynomial function on its support [0, c̄],
then the Equations (4)–(6) are polynomial equations and
we can apply the Gröbner bases methods.

Suppose the distribution of types is uniform on the interval
[0, 1], and so F(c) = c for c \in [0, 1]. Then Equations
(4)–(6) have the following form:

\begin{align*}
(c_H^∗ − c_L^∗)c_L^∗ &= (1 − c_H^∗)\mu, \quad (7) \\
[1 − 2(c_H^∗ − c_L^∗)]c_H^∗ &= c^Ld, \quad (8) \\
(1 − c_H^∗)(\mu − c^∗) + c_L^∗(−c^∗) &= F(c_L^∗)(−d). \quad (9)
\end{align*}

Using the simplified notation m = \mu and (c(3), c(2),
c(1)) = (c_H^∗, c^∗, c_L^∗), we write the system in SINGULAR as follows:

```plaintext
ring R = (0,m,d),(c(3),c(2),c(1)),lp;
option(redSB);
ideal I =((c(3)-c(1))*c(1)-(1-c(3))*m,
(1-2*(c(3)-c(1)))*c(3)-c(1)*d,
(1-c(3))*m-c(2)-c(1)*c(2)+c(1)*d));
ideal G=groebner(I);
```
SINGULAR produces the following output:

\[
G[1]=(-2^m+d-1)*c(1)**3+(2^m*d+m)*c(1)**2 + (m+1)*c(2)+(m-d)*c(1)+(m-d-m)
\]

\[
G[2]=(m+1)*c(2)+(m-d)*c(1)+(m-d-m)
\]

\[
G[3]=(-2^m+2^m)+c(3)+(2^m+d+1)*c(1)**2 + (-m^d)*c(1)+(2^m+2^m).
\]

With \( \bar{c} = 1 \) as the upper bound for the support of the distribution \( F \), the assumption \( \bar{c} < d \) implies \( d > 1 \). For sufficiently small \( m \) (\( \approx \mu \)) the univariate polynomial \( G[1] \) then has two sign changes, and so there can be at most two positive real solutions. For fixed values of \( d \) and \( m \) we can easily find all zeros of the Gröbner basis. Figure 5 displays the positive real solutions for \( d = 1.5 \) and small values of \( \mu \). (The third solution has negative real values for all three variables for \( \mu < 0.25 \).) There are three curves, one for each cutoff value \( c^L < c^* < c^H \). The respective upper branches of the curves for \( c^L \) and \( c^* \) together with the lower branch of the curve for \( c^H \) represent the equilibrium solution that is emphasized and deemed desirable by BS. (The three branches are marked with a point at \( \mu = 0.024 \).)

Note that, as in their lemma, \( c^H \to 0 \) as \( \mu \to 0. \) The respective other branches of the three curves represent the second equilibrium, which has the property that \( \bar{c} \to 1/2 \) as \( \mu \to 0 \). At \( \mu = 16/(143 + 19/\sqrt{57}) \approx 0.0558568 \) the two respective branches meet for each cutoff value. The univariate polynomial \( G[1] \) has a double solution \( c^L = 0.101588 \). As \( \mu \) increases further, the solutions become complex. For example, for \( \mu = 0.0559 \) the two solutions for \( c^L \) become \( 0.101611 \pm 0.0023494i \). The system (7)–(9) no longer has a positive real solution. This fact does not violate the lemma of BS because its claim holds only for sufficiently small \( \mu > 0 \).

We next show that the condition \( \bar{c} < d \) is important. When \( d = \bar{c} = 1 \), the Gröbner basis is sufficiently simple to allow for closed-form expressions in \( \mu \) for all three solutions of the system (7)–(9). The solution \((c^L, c^*, c^H) = (1, 1, 1)\) violates the inequalities (3). Figure 6 displays the remaining two solutions, which are

\[
\left(\frac{1}{4}(1 - \sqrt{1 - 8\mu}), \frac{1 + 7\mu - (1 - \mu)\sqrt{1 - 8\mu}}{4(1 + \mu)}, \frac{1}{2}\right).
\]

Figure 6. Equilibria for \( d = 1 \).

\[
\left(\frac{1}{4}(1 + \sqrt{1 - 8\mu}), \frac{1 + 7\mu + (1 - \mu)\sqrt{1 - 8\mu}}{4(1 + \mu)}, \frac{1}{2}\right).
\]

For \( \mu > 0.125 \), both solutions for \( c^L \) and \( c^* \) are complex. Note that \( c^H = 1/2 \), and so as \( \mu \to 0 \) it holds that \( c^H \neq 0 \).

Thus, the theorem of BS no longer holds for \( d = \bar{c} = 1 \).

This completes our analysis of the cheap-talk game. Note that in the analyzed range for \( \mu > 0 \) no coefficient of a leading term in the Gröbner basis ever has the value 0. The parameterized Gröbner basis specializes correctly for all analyzed parameter values. However, for \( \mu = 0 \) the Gröbner basis does not specialize because the linear term in \( c[3] \) has the coefficient 0 in \( G[3] \). A separate analysis shows that there is a regular solution at \((0, \mu, 0, 1/2)\) and a double solution at \((0, 0)\).

6. Multiple Steady States in OLG Models

Overlapping generations ("OLG") models are workhorses for both theoretical and applied analysis in economics (particularly in public finance), monetary theory, and macroeconomics. Robust examples of OLG economies with a continuum of competitive equilibria are well known in the economics literature. For example, Kehoe and Levine (1990) construct robust examples of realistically calibrated OLG models with agents living for three periods in which indeterminacy of equilibria occurs. They point out that the possibility of a continuum of competitive equilibria poses a serious challenge to applied equilibrium modeling in the spirit of the influential work by Auerbach and Kotlikoff (1987). As one possible escape from the indeterminacy of equilibria, Kehoe and Levine (1990) suggest focusing on stationary equilibria (steady states). Kehoe and Levine (1984) show that steady states are generally determinate. Clearly, the presence of multiple (albeit finitely many) steady states also poses problems to the application of OLG models in policy analysis. Unfortunately, conditions that ensure uniqueness of steady states are extremely restrictive, see Kehoe et al. (1991). Beyond these conditions, little if anything is known about multiplicity of steady-state equilibria in OLG models. We show how Gröbner bases methods can address this problem.
6.1. A Stationary OLG Model

We consider the so-called “double-ended infinity model” (Kehoe et al. 1991, Geanakoplos 2008) in which discrete time runs from minus infinity to plus infinity, \( t \in \mathbb{Z} = \{-\ldots,-2,-1,0,1,2,\ldots\} \). At each time \( t \) a representative agent is born and lives for \( N \geq 2 \) periods. Each period, those agents who are alive receive an endowment \( e_a \) that depends solely on their age, \( a = 1, \ldots, N \). An agent’s utility is time separable, with the utility of an agent born at time \( t \) given by

\[
U_t(c) = \sum_{a=1}^{N} u(c_a(t + a - 1)),
\]

where \( c_a(t + a - 1) \) denotes the consumption of an agent born at time \( t \) in period \( t + a - 1 \) (when he is in period \( a \) of his life). For simplicity and without loss of generality, we assume that agents do not discount and have time-invariant utility. The computational methods also apply to more general models.

A competitive equilibrium is defined as usual by market clearing and agent optimality, that is, it is given by a sequence of prices and consumption allocations \( (p(t), (c_a(t))_{a=1}^{N})_{t \in \mathbb{Z}} \) such that for each \( t \),

\[
\sum_{a=1}^{N} (\bar{c}_a(t) - e_a) = 0
\]

and

\[
(\bar{c}_1(t), \ldots, \bar{c}_N(t + N - 1)) \in \arg\max_{c(t), \ldots, c(t + N - 1)} U_t(c(t), \ldots, c(t + N - 1))
\]

subject to \( \sum_{a=1}^{N} p(t + a - 1)(c(t + a - 1) - e_a) = 0 \).

The computation of a general competitive equilibrium requires us to calculate infinitely many prices and consumption values. Clearly, our methods cannot compute arbitrary nonstationary equilibria. However, they allow us to find all stationary equilibria.

A steady state, or stationary equilibrium, is a collection of consumption allocations for all agents and all ages as well as prices, such that market clearing and agent optimality holds, and for all \( t \in \mathbb{Z} \),

\[
p_{t+1} = q > 0 \quad \text{and} \quad \bar{c}_a(t) = c_a.
\]

There are two types of such stationary equilibria. Kehoe and Levine (1984) prove that (generically) \( q = 1 \) corresponds to the unique “monetary” steady state and that there is an odd number of “real” (nonmonetary) steady states with \( q \neq 1 \).

This completes our concise description of steady states in our stationary OLG pure exchange economy. For detailed treatments of OLG economies, we refer the interested reader to Ljungqvist and Sargent (2004) and Geanakoplos (2008). These treatments relate our model to the classic OLG model of Samuelson (1958) and Gale (1973) with fiat money.

6.2. Polynomial Equilibrium System

In applied research, modelers often assume that agents’ per-period utility function is given by \( u(c) = c^{1-\sigma}/(1 - \sigma) \) for \( 0 < \sigma \neq 1 \) (and log utility for \( \sigma = 1 \)). For \( \sigma \in \mathbb{N} \), this utility function leads to polynomial equilibrium equations. Using the necessary and sufficient first-order conditions for agents’ utility maximization problems in addition to the market-clearing equations yields the following equilibrium system:

\[
c_{a+1} - c_a = 0, \quad a = 1, \ldots, N - 1,
\]

\[
\sum_{a=1}^{N} q^{a-1}(c_a - e_a) = 0,
\]

\[
\sum_{a=1}^{N} (c_a - e_a) = 0.
\]

Evidently, the system always has a solution with \( q = 1 \) and \( c_a = (1/N) \sum_{a=1}^{N} e_a \). This solution is the unique monetary steady state. It is called the “golden rule” steady state and is Pareto efficient. We now investigate the number of real steady states in this economy. For the purpose of finding all equilibria with SINGULAR, we define \( w = q^{1/\sigma} \) and rewrite the system of equilibrium equations as follows:

\[
c_{a+1}w - c_a = 0, \quad a = 1, \ldots, N - 1,
\]

\[
\sum_{a=1}^{N} w^{\sigma(a-1)}(c_a - e_a) = 0,
\]

\[
\sum_{a=1}^{N} c_a - e_a = 0.
\]

For integer-valued \( \sigma \) this system is polynomial. We do not examine the case of log utility because it is well known that there is a unique real steady state, see Kehoe et al. (1991). Instead, we begin our analysis with \( \sigma = 2 \) and subsequently examine larger levels of risk aversion. To give the reader some idea about the needed SINGULAR code and the resulting output, we show the code and output for \( N = 3 \) and \( \sigma = 2 \). As in §4, we use excess demand variables \( x(a) \) for \( c_a - e_a \). We denote the endowment parameters \( e_1, e_2, e_3 \) by \( e, f, g \), respectively.

\[
\text{int n = 4;}
\]
\[
\text{ring R= (0,e,f,g,b),x(1..n),lp;}
\]
\[
\text{option(redSB);}\]
\[
\text{ideal I = } -(f+x(2))*x(4)+(e+x(1)),\]
\[
-g*x(3)*x(4)+(f+x(2)),\]
\[
x(1)+x(2)*x(4)**2+x(3)**4, \ x(1)+x(2)+x(3));\]
\[
\text{ideal G=groebner(I);}
The resulting Gröbner basis is as follows.

\[ G[1] = (-g) x(4) + e + g \]  
\[ G[2] = (3e^2 + 3e^g + 3g^2) x(3) + (-e^2 x^2 + e x^2 + g^2 x^2 - 2x^2 - 2g^2 x^2 - 3g^2) x(3) \]  
\[ x(4) + (-e^3 x + e x^2 + g^2 x^2 - 2x^2 - 2g^2 x^2 - 3g^2) x(4) \]  
\[ G[3] = (-3e^2 x + 3e^2 x + 3g^2 x^2) x(2) + (-2e^2 x + g^2 x^2 + 3g^2 x^2 - 2x^2 - 2g^2 x^2 - 3g^2) x(2) \]

6.3. Uniqueness for \( \sigma = 2 \)

Singular computes the fully parameterized Gröbner basis in a few seconds for \( N = 3, 4, 5, 6 \), in a few minutes for \( N = 7, 8 \), and in about an hour for \( N = 9, 10 \). When \( N \) is even, the univariate representation is given by

\[ r(w) = \sum_{i=1}^{N} e_i w^{2(i-1)} - \left( \sum_{i=1}^{N} e_i \right) w^{N-1}. \]

For odd \( N \) the expression is

\[ r(w) = \sum_{i=1}^{(N-1)/2} e_i w^{2(i-1)} - \left( \sum_{i=1}^{(N-1)/2} e_i + \sum_{i=(N+3)/2}^{N} e_i \right) w^{N-1} + \sum_{i=(N+3)/2}^{N} e_i w^{2(i-1)}. \]

In both cases there are exactly two sign changes among the coefficients. Thus, Descartes’ Rule bounds the number of positive real solutions by two. However, we know that there must be at least two solutions. The golden rule monetary steady state is given by the solution \( w = 1 \). The remaining solution thus must be the only real steady state. To the best of our knowledge, this result was not previously known. In sum, direct Gröbner basis computation proves the following result.

**Proposition 1.** The double-ended infinity exchange economy with \( \sigma = 2 \) has a unique real steady state for all \( N \).

Applied researchers may consider a value of \( \sigma = 2 \) to be somewhat realistic for the calibrations of models. The proposition implies for the double-ended infinity model that researchers do not need to worry about multiplicity of real steady states for \( \sigma = 2 \). We next show that for integer-valued \( \sigma \geq 3 \) multiplicity of real steady states arises.

6.4. Larger Coefficients of Risk Aversion

For \( \sigma = 3 \) and \( N = 3 \) the univariate representation is given by

\[ r(w) = e_1 w^6 - (e_1 + e_2 + e_3) w^4 + (e_1 + 2e_2 + e_3) w^3 - (e_1 + e_2 + e_3) w^2 + e_1. \]

There are four sign changes, and so Descartes’ rule no longer guarantees the existence of a unique nonmonetary steady state. In fact, it is a simple exercise to construct economies with four equilibria, that is, one monetary and three real (nonmonetary) steady states. Figure 7 shows the positive real roots of the univariate representation \( r \) as a function of \( e_2 \) for \( e_1 = 1 \) and \( e_3 = \frac{1}{2} \). For \( e_2 < 10.575747 \) there is a unique real steady state in addition to the monetary steady state \((w = 1)\). For \( e_2 > 10.575747 \) the polynomial \( r \) has four positive real roots. All four solutions for \( w \) lead to positive consumption values for the \( N = 3 \) agents alive at any given time. Table 1 lists all four steady states for \( e_2 = 12 \).

The first two steady states are efficient equilibria with positive interest rates, the third steady state is the monetary steady state with a zero interest rate, and the fourth steady state is inefficient with a negative interest rate. Similar to the examples of indeterminacy of equilibria in Kehoe and Levine (1990), the existence of multiple real steady states requires substantial hump-shape in life-cycle income.

For \( \sigma = 3 \), unlike for the case of \( \sigma = 2 \), the number of sign changes in the univariate representation is no longer independent of \( N \). For example, for \( N = 5 \) (and

![Figure 7. Equilibria for \( e_1 = 1 \) and \( e_3 = \frac{1}{2} \).](image-url)
Table 2. Steady states for $e_2 = 12$.

<table>
<thead>
<tr>
<th>Eq</th>
<th>$w$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.479331</td>
<td>1.81485</td>
<td>3.78621</td>
<td>7.89894</td>
</tr>
<tr>
<td>2</td>
<td>0.775522</td>
<td>3.41586</td>
<td>4.40460</td>
<td>5.67953</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4.5</td>
<td>4.5</td>
<td>4.5</td>
</tr>
<tr>
<td>4</td>
<td>3.98751</td>
<td>10.2765</td>
<td>2.57718</td>
<td>0.646312</td>
</tr>
</tbody>
</table>

For $\sigma = 3$ we obtain a univariate representation with six sign changes. Denoting the endowment parameters $e_1, e_2, \ldots, e_5$ by $e, f, g, h, i$, respectively, the polynomial $G[1]$ is as follows:

$$G[1] = (-i)x(6)^{12} + (-h)x(6)^9$$

$$+ (e+f+g+h+i)x(6)^8 + (-e-f-g-h-i)x(6)^7$$

$$+ (e+f+g+h+i)x(6)^6 + (-e-f-g-h-i)x(6)^5$$

$$+ (e+f+g+h+i)x(6)^4 + (-e)x(6)^3 + (-e).$$

There are six sign changes, and so there could be up to six positive real solutions, and thus up to five real steady states in this economy.

Our last example of OLG economies outlines how we can apply the described methods to models used in applied work. Krueger and Kubler (2005) calibrate a nine-period OLG model to match observed U.S. data. Although they consider a stochastic model with capital accumulation and shocks to production, we can still use the labor income from their calibration as individual endowments because the calibration of life-cycle income is from income data. Figure 2 lists these endowments. For these endowments and $\sigma = 3$ the polynomial system of our double-ended infinity model has four real solutions. Two of them are not equilibria because prices are negative. Equilibrium consumption in the unique real steady state is

$$(2.10122, 1.80705, 1.55407, 1.33651, 1.1494, 0.988487, 0.850102, 0.73109, 0.62874),$$

and the interest rate is $(1/1.16279^3) - 1$. The interest rate is negative, and thus the equilibrium is inefficient. (We also computed equilibria for $\sigma = 4$ and $\sigma = 5$, and the real steady state remains unique.)

Clearly, the analysis of a realistically calibrated applied model is greatly aided by the existence of a unique steady state. Although Kehoe and Levine (1990) demonstrate that it is straightforward to construct fairly realistic examples with multiple steady states, many applied modelers hope (or even claim) that their computed (real) steady states are unique. Of course, it remains open for future research whether computed real steady states are in fact unique. The Gröbner basis methods offer an approach to examine this issue.

Table 2. Labor endowments.

<table>
<thead>
<tr>
<th>Age $\alpha$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j^a$</td>
<td>1</td>
<td>1.35</td>
<td>1.54</td>
<td>1.65</td>
<td>1.67</td>
<td>1.66</td>
<td>1.61</td>
<td>1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

7. Conclusion

Multiplicity of equilibria is a prevalent problem in many economic models. Often equilibria are characterized as solutions to a system of polynomial equations. Therefore, methods that allow the computation of all solutions to such systems are of great interest to economists.

In this paper we have provided an introduction to the application of Gröbner basis methods for finding all solutions to polynomial systems of equations. We have described a beautiful result from algebraic geometry, the Shape Lemma, which states under mild assumptions that a given polynomial system is equivalent to a much simpler system with identical solutions. The new system enables a fast and robust calculation of all solutions. Essentially, the computation of all solutions reduces to finding all roots of a single polynomial in a single unknown. We have described the software package SINGULAR, which computes the equivalent simple system. If all coefficients in the original equilibrium equations are rational numbers or parameters, then the computations of SINGULAR are exact. This fact implies that the described methods cannot only be used for a numerical approximation of equilibria, but in fact may allow us to prove theoretical results for the underlying economic model. Three economic applications have illustrated that without much prior knowledge in algebraic geometry, the described methods can be used to gain interesting insights into modern economic models.

In a companion paper (Kubler and Schmedders 2010), we prove that for large classes of general equilibrium models the assumption of semialgebraic marginal utility leads to polynomial equilibrium equations. We show that the Shape Lemma can be applied to a wide variety of GE models, and thereby build the theoretical foundation for a systematic analysis of multiplicity in applied general equilibrium.

Endnotes

2. Throughout this paper, we use the term generic loosely; if we focus on the reals it means for an open set of full measure.

Acknowledgments

The authors thank Gerhard Pfister for help with SINGULAR, and they are grateful to him and Bernd Sturmfels for patiently answering their questions on computational algebraic geometry. They gratefully acknowledge detailed comments on an earlier version by Ludwig Ensthaler, Philipp Renner, Ken Judd (coeditor of the special issue on computational economics), and an anonymous referee.

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