A Large Scale Optimization Model for Replicating Portfolios in the Life Insurance Industry

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Why do we need RPs

Solvency II sets out to establish new capital requirements, valuation techniques and governance and reporting standards.

Solvency Capital Requirements (SCR): Calculate fair values of insurance liabilities.

Life insurance liabilities are usually long term and their accurate valuation is complex.

Monte Carlo or Nested simulations attractive to assess guarantees and options within the life insurance contracts, e.g. profit sharing rules.

Simulation based approaches are time intense (typical valuation run takes up to 2h, 10,000 runs needed → 120 weeks ).
RP as a Life Insurance Valuation Tool

Approximation tool is required to represent life insurance liabilities in risk calculation → Replicating Portfolios (RPs).

RPs have a history in derivative pricing but are fairly recent used in the insurance industry.

RPs are a tool to reproduce a set of liability cash flows under different economic scenarios.

RP can be used for ALM and performance management, risk management, capital and value calculations and management information.
RP as a Life Insurance Valuation Tool

Two conditions must be satisfied to reap the benefits of an RP:

- RP must replicate liability cash flows for a large number of scenarios.
- Possibility to compute an RP fairly quickly.

RP process:

- Scenario generation for liabilities and candidate assets
- Optimization model to calibrate the candidate assets weights
- Tests to assess the quality of the RP
Literature on RP fairly limited:

Seemann (2009) provides a broad explanation of the complete RP process including optimization models based on the $L_2$ norm.

Burmeister et al. (2010) stress the advantages of a small RP and introduce a model with trading constraints.

Chen and Skoglund (2012) present an efficient linear programming approach to cash flow replication with mismatch constraints.
This Paper

Focus on the optimization and the testing of the RP:

- $L_1$ minimization model to calibrate candidate assets weights.
- Two linear reformulations of the model to solve the optimization for the whole time horizon in a single step.
- Introduction of several constraints.
- Several tests to assess the quality of the RP.
- Implementation of the model and the tests using three large scaled real-life data sets.
- Assessment of different RPs and the computational performance in GAMS.
Outline

Introduction
  Motivation and Summary

The Model
  Optimization Model

Tests
  OoS Scenario Set Movement, Market Value Movement, $R^2$

Model Specifications
  Scenario Sets, Assets and Parameter

Results
  A comparison of different RPs
  Detailed Results for three RPs
  Performance in GAMS

Conclusion
  Summary
The Basic RP Problem

Objective of an RP: Reproduce given liability cash flows with a pool of asset cash flows. Mathematically we want to solve:

\[
\begin{pmatrix}
ACF_{1,1} & ACF_{1,2} & \cdots & ACF_{1,A} \\
ACF_{2,1} & ACF_{2,2} & \cdots & ACF_{1,1} \\
\vdots & \vdots & \ddots & \vdots \\
ACF_{C,1} & \vdots & \cdots & ACF_{C,A}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_A
\end{pmatrix}
= 
\begin{pmatrix}
LCF_1 \\
LCF_2 \\
\vdots \\
LCF_C
\end{pmatrix}
\]

Problem (1) does not have a solution (more scenarios than assets).

Solution: We match cash flows as closely as possible using the $L_1$ norm.
Notation

Finite-horizon model in discrete time, \( t = 0, 1, 2, \ldots, T \)

Scenario sets, \( k = 1, 2, \ldots, K \)

Base scenario set, \( k = 1 \), Non-base scenario sets, \( k = 2, 3, \ldots, K \)

Finite number of scenarios per scenario set, \( s = 1, 2, \ldots, S \)

Assets classes, \( a = 1, 2, \ldots, A \), with a set of candidate assets, \( i = 1, 2, \ldots, l_a \)
Objective

Replication of the discounted liability cash flows with a set of candidate assets using the $L_1$ norm.

\[
\min_{\mathbf{x}} \sum_{k=1}^{K} \sum_{s=1}^{S} \sum_{t=1}^{T} |LCF_{k,s,t} - \sum_{a=1}^{A} \sum_{i=1}^{I_a} ACF_{k,s,t,a,i}(x_{a,i})| \tag{2}
\]

Decision variables $x_{a,i}$ are the holdings in the candidate assets.

$LCF_{k,s,t}$ are the discounted liability cash flows.

$ACF_{k,s,t,a,i}$ are the discounted asset cash flows.
Linear Model 1

Generally used in RP and related fields involving $L_1$-approximations.

We introduce the set of auxiliary variables $z_{k,s,t}$.

\[
\min_{l,s,z} \sum_{k=1}^{K} \sum_{s=1}^{S} \sum_{t=1}^{T} z_{k,s,t} \quad (3)
\]

\[
z_{k,s,t} \geq LCF_{k,s,t} - \sum_{a=1}^{A} \sum_{i=1}^{I_a} ACF_{k,s,t,a,i}(x_{a,i}) \quad (4)
\]

\[
- z_{k,s,t} \leq LCF_{k,s,t} - \sum_{a=1}^{A} \sum_{i=1}^{I_a} ACF_{k,s,t,a,i}(x_{a,i}) \quad (5)
\]
Linear Model 2

Widely used in portfolio optimization, but not for RPs.

We introduce the set of auxiliary variables $y_{k,s,t}^+$ and $y_{k,s,t}^-$.

\[
\min_{l,s,y^+,y^-} \sum_{k=1}^{K} \sum_{s=1}^{S} \sum_{t=1}^{T} (y_{k,s,t}^+ + y_{k,s,t}^-) \tag{6}
\]

\[
y_{k,s,t}^+ - y_{k,s,t}^- = LCF_{k,s,t} - \sum_{a=1}^{A} \sum_{i=1}^{I_a} ACF_{k,s,t,a,i}(x_{a,i}) \tag{7}
\]

\[
y_{k,s,t}^+, y_{k,s,t}^- \geq 0 \tag{8}
\]
Additional Constraints - Notionals

Avoid offsetting problems and limit the total number of assets in the RP.

Separate decision variables $x_{a,i}$ into long position ($l_{a,i}$) and short positions ($s_{a,i}$):

$$x_{a,i} = l_{a,i} - s_{a,i}$$  \hspace{1cm} (9)

$$l_{a,i}, s_{a,i} \geq 0$$  \hspace{1cm} (10)

$$nl\left(\sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{1,s,t}/S\right) \geq \sum_{i=1}^{l_a}(l_{a,i} + s_{a,i})nv_{a}$$  \hspace{1cm} (11)

should hold up for $a = 1, 2, \ldots, A$

$n l$ is the notional limit, $nv_{a}$ is the notional value of asset class $a$. 
Additional Constraints - Scenario Set Movements

Provide a good match for all kind of future events → limit scenario set movements.

\[
adub \geq \left( \sum_{s=1}^{S} \sum_{t=1}^{T} \sum_{a=1}^{A} \sum_{i=1}^{I_a} ACF_{k,s,t,a,i}(l_{a,i} - s_{a,i}) - \sum_{s=1}^{S} \sum_{t=1}^{T} \sum_{a=1}^{A} \sum_{i=1}^{I_a} ACF_{1,s,t,a,i}(l_{a,i} - s_{a,i}) \right)
- \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{k,s,t} + \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{1,s,t} / \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{1,s,t} \right) \tag{12}
\]

\[
adlb \leq \left( \sum_{s=1}^{S} \sum_{t=1}^{T} \sum_{a=1}^{A} \sum_{i=1}^{I_a} ACF_{k,s,t,a,i}(l_{a,i} - s_{a,i}) - \sum_{s=1}^{S} \sum_{t=1}^{T} \sum_{a=1}^{A} \sum_{i=1}^{I_a} ACF_{1,s,t,a,i}(l_{a,i} - s_{a,i}) \right)
- \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{k,s,t} + \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{1,s,t} / \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{1,s,t} \right) \tag{13}
\]

should hold up for \( k = 1, 2, \ldots, K \)

upper (\( adub \)) and lower (\( adlb \)) bound for scenario set difference.
Trimming

Theoretically several optimal solutions exist for the two optimization models → we want a solution with $l_{a,i} s_{a,i} = 0$.

Trimmed solution $(z, \bar{l}, \bar{s})$:

$$\bar{l}_{a,i} = l_{a,i} - \delta_{a,i} \quad \text{for all } i = 1, \ldots, I_a \text{ and } a = 1, \ldots, A.$$  \hspace{1cm} (14)

$$\bar{s}_{a,i} = s_{a,i} - \delta_{a,i} \quad \text{for all } i = 1, \ldots, I_a \text{ and } a = 1, \ldots, A.$$  \hspace{1cm} (15)

where $\delta_{a,i}$ is computed as:

$$\delta_{a,i} = \min\{l_{a,i}, s_{a,i}\} \quad \text{for all } i = 1, \ldots, I_a \text{ and } a = 1, \ldots, A.$$  \hspace{1cm} (16)
Out-of-Sample Scenario Set Movement

Similar behavior of discounted asset cash flows and discounted liability cash flows for all non base scenarios compared to the base scenario using out-of-sample data.

\[
Oos_k = \left( \sum_{s=1}^{S} \sum_{t=1}^{T} \sum_{a=1}^{A} \sum_{i=1}^{l_a} ACF_{k,s,t,a,i}^{os}(l_{a,i} - s_{a,i}) \right) - \sum_{s=1}^{S} \sum_{t=1}^{T} \sum_{a=1}^{A} \sum_{i=1}^{l_a} ACF_{1,s,t,a,i}^{os}(l_{a,i} - s_{a,i}) \\
\quad - \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{k,s,t}^{os} + \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{1,s,t}^{os}) / \sum_{s=1}^{S} \sum_{t=1}^{T} LCF_{1,s,t}^{os}
\]

for all non base scenario sets \( k = 2, 3, \ldots, K \)

\( LCF_{k,s,t} \) are the discounted liability cash flows.

\( ACF_{k,s,t,a,i} \) are the discounted asset cash flows.
Market Value Movement

Similar behavior of market value movements for all non-base market value scenario sets compared to the base market value scenario set.

Market value scenario sets, \( m = 1, \ldots, M \), base scenario set, \( m = 1 \), non-base scenario sets \( m = 2, 3, \ldots, M \).

\[
MV_m = \left( \sum_{a=1}^{A} \sum_{i=1}^{l_a} MVA_{m,a,i}(l_{a,i} - s_{a,i}) - MVA_{1,a,i}(l_{a,i} - s_{a,i}) - (MVL_m - MVL_1) \right) / MVL_1
\]

for all non base market value scenario sets \( m = 2, 3, \ldots, M \)

\( MVA_{1,a,i} \) and \( MVA_{m,a,i} \) are the asset market values. \( MVL_1 \) and \( MVL_m \) are the liability market values.
The coefficient of Determination - $R^2$

In-sample $R^2$ and Out-of-Sample $R^2$:

$$R^2_{is} = \frac{\sigma(LCF, ACF)^2}{\sigma^2(LCF)\sigma^2(ACF)}$$  \hspace{1cm} (19)$$

$$R^2_{os} = \frac{\sigma(LCF^{os}, ACF^{os})^2}{\sigma^2(LCF^{os})\sigma^2(ACF^{os})}$$  \hspace{1cm} (20)$$
Specific Problem

Time horizon, $T = 40$

13 scenario sets, $K = 13$, with 100 scenarios each, $S = 100$

15 market value scenario sets, $M = 15$

5 asset classes, $A = 5$, with up to $I_a = 720$ candidate assets
# Scenario Set Overview

## Table: List of Scenario Sets

<table>
<thead>
<tr>
<th>Scenario Set</th>
<th>Notation</th>
<th>Feature</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$k = m = 1$</td>
<td>base</td>
</tr>
<tr>
<td>2</td>
<td>$k = m = 2$</td>
<td>Yield curve up small</td>
</tr>
<tr>
<td>3</td>
<td>$k = m = 3$</td>
<td>Yield curve down small</td>
</tr>
<tr>
<td>4</td>
<td>$k = m = 4$</td>
<td>Yield curve up big</td>
</tr>
<tr>
<td>5</td>
<td>$k = m = 5$</td>
<td>Yield curve down big</td>
</tr>
<tr>
<td>6</td>
<td>$k = m = 6$</td>
<td>Equity index down big</td>
</tr>
<tr>
<td>7</td>
<td>$k = m = 7$</td>
<td>Property index down big</td>
</tr>
<tr>
<td>8</td>
<td>$k = m = 8$</td>
<td>Equity index down small</td>
</tr>
<tr>
<td>9</td>
<td>$k = m = 9$</td>
<td>Property index down small</td>
</tr>
<tr>
<td>10</td>
<td>$k = m = 10$</td>
<td>Interest rate volatility up</td>
</tr>
<tr>
<td>11</td>
<td>$k = m = 11$</td>
<td>Interest rate volatility down</td>
</tr>
<tr>
<td>12</td>
<td>$k = m = 12$</td>
<td>Equity index volatility up</td>
</tr>
<tr>
<td>13</td>
<td>$k = m = 13$</td>
<td>Equity index volatility down</td>
</tr>
<tr>
<td>14</td>
<td>$m = 14$</td>
<td>Shock 1</td>
</tr>
<tr>
<td>15</td>
<td>$m = 15$</td>
<td>Shock 2</td>
</tr>
</tbody>
</table>
## Asset Universe

**Table : Available Asset Universe**

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Notation</th>
<th>Available Assets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Coupon Bonds</td>
<td>$a = 1$</td>
<td>$I_1 = 40$</td>
</tr>
<tr>
<td>Equity Property Indexes</td>
<td>$a = 2$</td>
<td>$I_2 = 80$</td>
</tr>
<tr>
<td>Cash Indexes</td>
<td>$a = 3$</td>
<td>$I_3 = 40$</td>
</tr>
<tr>
<td>Interest Rate Swap Payer Asset</td>
<td>$a = 4$</td>
<td>$I_4 = 39$</td>
</tr>
<tr>
<td>Pay Fixed Swaptions</td>
<td>$a = 5$</td>
<td>$I_5 = 720$</td>
</tr>
</tbody>
</table>
# Model and Test Parameters

## Table : Parameter Overview

<table>
<thead>
<tr>
<th>Optimization Model - Provided Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$LCF_{k,s,t}$</td>
</tr>
<tr>
<td>$ACF_{k,s,t,a,i}$</td>
</tr>
<tr>
<td>$n v_5 = 100$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Optimization Model - Determined Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n l = 3$</td>
</tr>
<tr>
<td>$adub$</td>
</tr>
<tr>
<td>$adlb$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Tests - Out-of-Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>$LCF_{k,s,t}^{os}$</td>
</tr>
<tr>
<td>$ACF_{k,s,t,a,i}^{os}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Tests - Market Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MVL_m$</td>
</tr>
<tr>
<td>$MVA_{m,a,i}$</td>
</tr>
</tbody>
</table>
Overview

We ran one UC optimization without additional constraints (11), (12) and (13).

We ran 17 optimizations with absolute allowed differences between 0.8% and 0.1%.

We report relative $L_1$ errors:

$$RL_1 = \frac{z_{k,s,t}}{L_{1\text{UC}}} = \frac{y_{k,s,t}^+ + y_{k,s,t}^-}{L_{1\text{UC}}}$$

(21)
## Table: Results for different RPs

<table>
<thead>
<tr>
<th>AAD</th>
<th>RL&lt;sub&gt;1&lt;/sub&gt;</th>
<th>MV Mov</th>
<th>Oos SS Mov</th>
<th>R&lt;sup&gt;2&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Is</td>
<td>Oos</td>
<td>Av</td>
<td>WC</td>
</tr>
<tr>
<td>UC</td>
<td>1.000</td>
<td>1.037</td>
<td>0.676%</td>
<td>2.500%</td>
</tr>
<tr>
<td>0.8%</td>
<td>1.018</td>
<td>1.027</td>
<td>0.616%</td>
<td>2.089%</td>
</tr>
<tr>
<td>0.75%</td>
<td>1.019</td>
<td>1.028</td>
<td>0.600%</td>
<td>2.035%</td>
</tr>
<tr>
<td>0.7%</td>
<td>1.020</td>
<td>1.030</td>
<td>0.589%</td>
<td>1.996%</td>
</tr>
<tr>
<td>0.65%</td>
<td>1.023</td>
<td>1.033</td>
<td>0.572%</td>
<td>1.941%</td>
</tr>
<tr>
<td>0.6%</td>
<td>1.028</td>
<td>1.038</td>
<td>0.556%</td>
<td>1.914%</td>
</tr>
<tr>
<td>0.55%</td>
<td>1.037</td>
<td>1.048</td>
<td>0.538%</td>
<td>1.890%</td>
</tr>
<tr>
<td>0.5%</td>
<td>1.050</td>
<td>1.061</td>
<td>0.504%</td>
<td>1.811%</td>
</tr>
<tr>
<td>0.45%</td>
<td>1.069</td>
<td>1.080</td>
<td>0.463%</td>
<td>1.741%</td>
</tr>
<tr>
<td>0.4%</td>
<td>1.092</td>
<td>1.104</td>
<td>0.421%</td>
<td>1.665%</td>
</tr>
<tr>
<td>0.35%</td>
<td>1.121</td>
<td>1.136</td>
<td>0.382%</td>
<td>1.597%</td>
</tr>
<tr>
<td>0.3%</td>
<td>1.153</td>
<td>1.173</td>
<td>0.348%</td>
<td>1.529%</td>
</tr>
<tr>
<td>0.25%</td>
<td>1.190</td>
<td>1.212</td>
<td>0.325%</td>
<td>1.467%</td>
</tr>
<tr>
<td>0.225%</td>
<td>1.210</td>
<td>1.235</td>
<td>0.327%</td>
<td>1.436%</td>
</tr>
<tr>
<td>0.2%</td>
<td>1.230</td>
<td>1.257</td>
<td>0.322%</td>
<td>1.401%</td>
</tr>
<tr>
<td>0.175%</td>
<td>1.251</td>
<td>1.279</td>
<td>0.319%</td>
<td>1.367%</td>
</tr>
<tr>
<td>0.15%</td>
<td>1.320</td>
<td>1.340</td>
<td>0.332%</td>
<td>1.323%</td>
</tr>
<tr>
<td>0.1%</td>
<td>1.614</td>
<td>1.589</td>
<td>0.382%</td>
<td>1.281%</td>
</tr>
</tbody>
</table>
Effects of the absolute allowed Scenario Set Difference

Reduction of the absolute allowed scenario set difference leads to:

- An $L_1$-error increase.
- A decrease in market value movements of the worst case scenario set.
- A decrease in average market value movements until we reach an absolute allowed scenario set difference of 0.175%.
Effects of the absolute allowed Scenario Set Difference

Searching for an ideal RP we face several trade-offs:

- Small market value movements vs. small relative $L_1$-error.
- Small Out-of-sample scenario set movements vs. small relative $L_1$-error.
Total Coefficient of Determination - $R^2_{Is}$ and $R^2_{Oos}$

Trade-off between average absolute market value movements and correlation, DM and DSD:

- Smallest average market value movement comes with $R^2_{Is}$ of 98.1%.
- Smallest average market value movement comes with $R^2_{Oos}$ of 97.8%.
Detailed Results for three RPs

We analyze the RPs with an AAD of 0.175%, 0.25% and 0.45%:

- RP with an AAD of 0.175% has the smallest average market value movement.
- RP with an AAD of 0.25% has the smallest average Oos scenario set movement.
- RP with an AAD of 0.45% has the smallest worst case Oos scenario set movement.
Composition

Notional values of the three RPs:

- Notional values are 300% for pay fixed swaptions and smaller than 300% for the four other asset classes.
- Notional values increase when the AAD decreases.
Composition

Total assets in the RP for three different AAD

- Number of selected candidate assets decreases when the AAD decreases. 415 (AAD of 0.45%), 406 (AAD of 0.25%), 395 (AAD of 0.175%) out of 919 candidate assets are selected for the RP.
- All candidate assets are selected in the first four asset classes.
• Highest absolute market value movement for scenario set 14.
• All other scenario sets have absolute market value movements smaller than 0.7%.
The out-of-sample scenario set movement is smaller than 0.8% for all scenario sets for each of the three RPs.
The RP with an AAD of 0.45% has the best annual $R^2_{Oos}$ in every year.

Annual $R^2_{Oos}$ troubling for years three and ten, especially for the RP with an AAD of 0.175%.
## Model Statistics

**Table**: Comparison of GAMS Model Statistics

<table>
<thead>
<tr>
<th></th>
<th>Linear Model 1</th>
<th>Linear Model 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks of Equations</td>
<td>50</td>
<td>42</td>
</tr>
<tr>
<td>Single Equations</td>
<td>156,026</td>
<td>104,026</td>
</tr>
<tr>
<td>Blocks of Variables</td>
<td>33</td>
<td>41</td>
</tr>
<tr>
<td>Single Variables</td>
<td>105,839</td>
<td>157,839</td>
</tr>
<tr>
<td>Non Zero Elements</td>
<td>3,582,791</td>
<td>2,574,341</td>
</tr>
</tbody>
</table>
• Running times are significantly different between the two models and the three methods.
• Linear model 2 with Sifting has the best running times.
Assumptions and Limitations

Implementation depends on the quality of the scenario generation data provided to us.

We use a predefined set of candidate assets.

We do not weigh the RP quality criteria.

Year 3 has troubling values for $R^2_{Oos}$. 
Summary

Large scale $L_1$ minimization model in the life insurance industry:

- The two linear formulations enable us to solve the model in a single step and the introduction of several constraints.
- Implementation with real life data shows that suitable RPs with a good fit can be found with the model.
- Trade offs between certain RP quality criteria.
- Linear model 2 has advantageous features for RP problems and is superior to linear model 1.
- Combination of linear model 2 and the sifting method has by far the best running times.