Abstract

This paper presents an analysis of the higher-order dynamics of key financial quantities in asset-pricing models with recursive preferences. For this purpose, we first describe a projection-based algorithm for solving such models. The method outperforms common methods like discretization and log-linearization in terms of efficiency and accuracy. Our algorithm allows us to document the presence of strong nonlinear effects in the modern long-run risks models which cannot be captured by the common methods. For example, for a prominent recent calibration of a popular long-run risks model, the log-linearization approach overstates the equity premium by 100 basis points or 22.5%. The increasing complexity of state-of-the-art asset-pricing models leads to complex nonlinear equilibrium functions with considerable curvature which in turn have sizable economic implications. Therefore, these models require numerical solution methods, such as the projection methods presented in this paper, that can adequately describe the higher-order equilibrium features.

Keywords: Asset pricing, discretization, log-linearization, nonlinear dynamics, projection methods.

JEL codes: G11, G12.

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1 Introduction

This paper presents an analysis of the higher-order dynamics of key financial quantities in asset-pricing models with recursive preferences. For this purpose, we first describe a projection-based algorithm for solving such models. The method outperforms common methods like discretization and log-linearization in terms of efficiency and accuracy. Our algorithm allows us to document the presence of strong nonlinear effects in the modern long-run risks models which cannot be captured by the common methods. For example, for the recent calibration by Bansal, Kiku, and Yaron (2012a) of the influential Bansal and Yaron (2004) long-run risks model, the log-linearization approach overstates the equity premium by 100 basis points or 22.5%. The increasing complexity of state-of-the-art asset-pricing models leads to complex nonlinear equilibrium functions with considerable curvature, which has sizable economic implications. Therefore, a correct analysis of these models requires numerical solution methods, such as the projection methods presented in this paper, that can adequately describe the higher-order equilibrium features.

Asset-pricing models have become increasingly complex over the last three decades. The first generation of such models, developed in the 1980s (Grossman and Shiller (1981), Hansen and Singleton (1982), Mehra and Prescott (1985)), proved inadequate in explaining large-scale features of financial markets, such as the high equity premium and the low risk-free rate. As the literature on asset-pricing evolved and matured over time, researchers added more and more complex features to their models with such as, among others, incomplete markets in form of uninsurable income risks, frictions such as borrowing or collateral constraints, time-varying risk aversion, and heterogenous expectations. While these additional features had varying degrees of success, recently the new generation of long-run risks models (e.g. Bansal and Yaron (2004) or Hansen, Heaton, and Li (2008)) with their interplay of long-run risks, stochastic volatility, and recursive preferences have had considerably more success in resolving long-standing asset pricing puzzles.

These new models feature both highly nonlinear preferences structures as well as complex specifications for the exogenous driving forces of the economy. To handle the complexity, researchers must resort to approximation to make their models tractable. In their original long-run risks paper, Bansal and Yaron (2004) introduce a log-linear approximation of the price-dividend ratio to derive the model’s asset-pricing implications. Traditional log-linearization (see Judd (1996) for a careful derivation) solves the model in terms of very small deviations from the deterministic steady state. It is well-known that both time-varying volatility and Epstein-Zin risk aversion make no contribution to the log-linearized solution (Caldara,
Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012)), and that even for CRRA utility the approximation errors using standard log-linearizations are quite large (Collard and Juillard (2001)). Therefore new techniques have been developed that linearize around the risk-adjusted steady state of the model (see for example Juillard (2011), de Groot (2013) or Meyer-Gohde (2014)). Chen, Cosimano, and Himonas (2014, Section 3.4) provides a recent survey of log-linearized approximations. Bansal and Yaron (2004) proceed along this line, and linearize the model’s pricing kernel. This does not suffer from the same problems as the traditional method, and the bulk of the long-run risks literature (Bansal, Kiku, and Yaron (2010), Bansal, Kiku, and Yaron (2012a), Bollerslev, Tauchen, and Zhou (2009), Kaltenbrunner and Lochstoer (2010), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), Bansal and Shaliastovich (2013), Constantinides and Ghosh (2011), Bansal, Kiku, Shaliastovich, and Yaron (2014) or Beeler and Campbell (2012), among others) have followed the original Bansal-Yaron paper in using this approximation to solve the model.

By its very nature, a log-linear approximation will miss higher-order effects. Do these higher-order effects matter? We show that the errors introduced by this approximation can be large and economically significant. For example, using the parameters in Bansal, Kiku, and Yaron (2012a) we find that the log-linearization overestimates the equity premium by a percentage point (the true equity premium for the model is 4.7%, while the one computed by the log-linearization is 5.8%, an overestimate of 22.5%). The log-linearization also overestimates the volatility of the price-dividend ratio by a significant fraction (the true model solution is 0.24, while the log-linearized solution gives you 0.30, an overestimate of again about 22.5%).

So if even the Bansal-Yaron log-linearization has the potential to introduce large errors, what should we use instead? We consider two alternative families of methods. One method is to replace the continuous state space by a finite-state Markov chain and solve the discrete model. Mehra and Prescott (1985), in their original paper, use a two-state Markov chain. More general methods were introduced by Tauchen (1986) and Tauchen and Hussey (1991), and have been applied by many subsequent works, such as Guvenen (2009), Heaton and Lucas (1996), Heaton and Lucas (2000), Hansen, Heaton, and Yaron (1996), and Campbell (1993). We show that these methods are not adequate for the task at hand, and in fact can provide worse approximations than the Bansal-Yaron log-linearization for the newest generation of asset-pricing models, unless the number of nodes is very large. For example, in the recent calibration of the long-run risks model by Bansal, Kiku, and Yaron (2012a), using even 50 nodes for each state variable is insufficient to accurately approximate the second-order moments for asset prices, while it already takes several hours to compute solutions.

In contrast, projection methods (Judd (1992)) work very well. Projection methods, which ultimately derive from numerical methods to solve partial differential equations in physics,
are conceptually more complex than log-linearization or discretization techniques. We find, however, that they work extremely well in solving general asset-pricing models with recursive preference structures and complex exogenous processes, and that surprisingly low-dimensional approximations provide high-levels of accuracy. (See Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012) for similar success in the stochastic growth case.) The method is also robust to changes in model parameters. For example, in the long-run risks model the log price-dividend ratio becomes increasingly nonlinear as persistence increases, so the Bansal-Yaron approximation performs acceptably for some parameters (such that those in Bansal and Yaron (2004)), but breaks down for others (Bansal, Kiku, and Yaron (2012a)).

The paper is organized as follows. Section 2 describes the general solution algorithm used in this paper. Afterwards we apply the method to two asset-pricing models and compare it to commonly used methods in the literature. For this we first consider the endowment economy of Tallarini (2000) in Section 3. The exercise serves to analyze the factors that drive the accuracy of the different approximation methods and to understand, why and in which cases the methods fail to compute accurate solutions. Afterwards we provide a full evaluation for the long-run risks model of Bansal and Yaron (2004) in Section 4. Section 5 concludes.

## 2 Solution Method

We consider a standard asset-pricing model with a representative agent and recursive preferences as in Epstein and Zin (1989) and Weil (1989). Indirect utility at time $t$, $V_t$, is given recursively as

$$V_t = \left(1 - \delta\right)C_t^{1-\gamma} + \delta \left[E_t \left(V_{t+1}^{1-\gamma}\right)^{\frac{\theta}{\psi}}\right]^{\frac{1}{1-\gamma}}. \tag{1}$$

In this parametrization, $C_t$ is consumption, $\delta$ is the time discount factor, $\gamma$ determines the level of relative risk aversion, $\psi$ is the intertemporal elasticity of substitution, and $\theta = \frac{1}{1-\gamma}$. For $\theta = 1$ the agent has standard CRRA preferences and $\theta < 1$ indicates a preference for the early resolution of risk. Using the agent’s Euler equation, Epstein and Zin (1989) show that the gross return of asset $i$, $R_{i,t+1} = \frac{P_{i,t+1}D_{i,t+1}}{P_{i,t}}$, must satisfy the following pricing equation,

$$E_t \left[\delta^\theta \left(C_{t+1}^{1-\gamma}/C_t\right)^{-\frac{\theta}{\psi}} P_{w,t+1}^{\theta-1} R_{i,t+1}\right] = 1. \tag{2}$$

The term $R_{w,t+1}$ denotes the return on a claim to aggregate consumption, $R_{w,t+1} = \frac{W_{t+1}}{W_t - C_t}$, where $W_t$ is the (unobserved) wealth level of the agent at time $t$. As equation (2) has to hold for all assets $i$, it must also hold for the return of the aggregate consumption claim. So $R_{w,t+1}$...
is determined by the wealth-Euler equation given by

\[
E_t \left[ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{w,t+1}^\theta \right] = 1.
\] (3)

The objective of this paper is to analyze the dynamic behavior of the equilibrium asset returns that are determined by the conditions (2) and (3). For this purpose we need a highly accurate solution method which is capable to correctly capture higher-order features of the asset returns. Therefore, we next describe an algorithm using projection methods to solve the general asset-pricing model given by equations (2) and (3). Appendix A provides a generic description of the projection methods used in this paper. Projection methods are a general-purpose tool for solving functional equations. They were first introduced by physicists and engineers to solve partial differential equations, but they can be used to solve the types of fixed-point equations that arise in economics. (See Judd (1992) for an introduction to projection methods for economists.) In the following we describe how to apply the general projection method to the asset-pricing model given by equations (2) and (3). Our description does not strive for maximal generality but instead is meant to simply convey the key steps of applying a projection approach to the asset-pricing model.

2.1 Projection Methods Applied to Asset-Pricing Models

Projection methods (see Judd (1992) or Chen, Cosimano, and Himonas (2014)) are a general tool for solving functional equations of the form

\[(Gz)(x) = 0, \quad \forall x\]

(4)

where the variable \(x\) resides in a (state) space \(X \subset \mathbb{R}^l, l \geq 1\), and \(z\) is an unknown function with domain \(X\), so \(z: X \rightarrow \mathbb{R}^m\). The given operator \(G\) is a continuous mapping between two function spaces.

To apply a projection method to the asset-pricing model, we express the equilibrium conditions as a functional equation of the type (4). For this purpose, we need to choose an appropriate state space and perform the usual transformation from an equilibrium described by infinite sequences (with a time index \(t\)) to the equilibrium being described by functions of some state variables(s) \(x\) on a state space \(X\). We denote the current state of the economy by \(x\) and the subsequent state in the next period by \(x'\). (For example in the original model by Mehra and Prescott (1985), the state \(x\) is log consumption growth and \(X \subset \mathbb{R}^1\); in the model of Bansal and Yaron (2004), the state \(x\) consists of the long-run mean of consumption growth.
(denoted by \(x_t\) in that paper) and the variance of consumption growth (denoted by \(\sigma^2\)), so \(X \subset \mathbb{R}^2\). We assume that the probability distribution of next period’s state \(x’\) conditional on the current state \(x\) is defined by a density \(f_x\).

First note that we solve the model in two steps. In the first step, we use the projection method to solve the wealth-Euler equation (3) to obtain the return on wealth. Once the return on wealth is known, then, in a second step, we can solve for any asset return by applying the projection approach to equation (2). For the first step, write equation (3) in state-space representation

\[
E \left[ \exp \left( \theta \log \delta - \frac{\theta}{\psi} \Delta c(x'|x) + \theta r_w(x'|x) \right) \bigg| x \right] = 1, \quad \forall x,
\]

where lower case letters denote logs of variables and \(\Delta c(x'|x) = c(x') - c(x)\). We write the model in logs, because the function we solve for is the log wealth-consumption ratio \(z_w(x) = \log \left( \frac{W(x)}{C(x)} \right)\). Next, write the state-dependent log return of the aggregate consumption claim as

\[
r_w(x'|x) = \log \left( \frac{W(x')}{W(x) - C(x)} \right) = \log \left( \frac{W(x')}{\frac{W(x)}{C(x)} - 1} \times \frac{C(x')}{C(x)} \right) = z_w(x') - \log \left( e^{z_w(x)} - 1 \right) + \Delta c(x'|x).
\]

Inserting the last term in equation (5) yields

\[
E \left[ \exp \left( \theta \left( \log \delta + (1 - \frac{1}{\psi}) \Delta c(x'|x) + z_w(x') - \log \left( e^{z_w(x)} - 1 \right) \right) \right) - 1 \bigg| x \right] = 0, \quad \forall x.
\]

Equivalently,

\[
0 = \int_X \left[ \exp \left( \theta \left( \log \delta + (1 - \frac{1}{\psi}) \Delta c(x'|x) + z_w(x') - \log \left( e^{z_w(x)} - 1 \right) \right) \right) - 1 \right] df_x
\]

which is a functional equation of the form (4) and allows us to apply the projection approach.

The unknown solution function to this equilibrium condition, \(z_w\), is an element of a function space which is an infinite-dimensional vector space. A key feature of every projection method is to approximate the solution function \(z_w\) by an element from a finite-dimensional space. Specifically, we use the approximation \(\hat{z}_w(x; \alpha_w) = \sum_{k=0}^{n} \alpha_{w,k} \Lambda_k(x)\), where \(\{\Lambda_k\}_{k \in \{0, 1, \ldots, n\}}\) is a set of chosen (known) basis functions and \(\alpha_w = [\alpha_{w,0}, \alpha_{w,1}, \ldots, \alpha_{w,n}]\) are unknown coefficients. Replacing the exact solution \(z_w(x)\) by the approximation \(\hat{z}_w(x; \alpha_w)\) leads us to the residual
function $\hat{F}_w$ for the rearranged wealth-Euler equation (8), which is defined by

$$\hat{F}_w(x; \alpha_w) = \int_X \left[ \exp \left( \theta \left( \log \delta + (1 - \frac{1}{\psi}) \Delta c(x'|x) + \hat{\gamma}_w(x') - \log \left( e^{\hat{\gamma}_w(x)} - 1 \right) \right) \right) - 1 \right] df_x. \quad (9)$$

We can determine values for the unknown solution coefficients $\alpha_w$ by imposing a projection condition on the residual term $\hat{F}_w(x; \alpha_w)$. In this paper we employ two different such projection conditions, the collocation and the Galerkin method, see Appendix A. The values for the coefficients $\alpha_w$ determine the state-dependent wealth-consumption ratio $\hat{\gamma}_w(x; \alpha_w)$ which in turn leads to the (approximate) return function of the aggregate consumption claim, $\hat{r}_w(x'|x; \alpha_w) = \hat{\gamma}_w(x'; \alpha_w) - \log \left( e^{\hat{\gamma}_w(x; \alpha_w)} - 1 \right) + \Delta c(x'|x)$.

With $\hat{r}_w(x'|x; \alpha_w)$ at hand, we can now develop an approach to compute the return of any asset $i$ using equation (2). Analogous to the first step, we solve for the log price dividend ratio $z_i(x) = \log \left( \frac{p_i(x)}{D_i(x)} \right)$ and rewrite the state-dependent log return of asset $i$ as

$$r_i(x'|x) = \log \left( \frac{p_i(x') + D_i(x')}{P_i(x)} \right) = \log \left( \frac{p_i(x') + 1}{P_i(x)} \times \frac{D_i(x')}{D_i(x)} \right) = \log \left( e^{z_i(x')} + 1 \right) - z_i(x) + \Delta d_i(x'|x). \quad (10)$$

Writing the Euler equation (2) in state-space representation and formulating it in logs yields

$$E \left[ \exp \left( \theta \log \delta - \frac{\theta}{\psi} \Delta c(x'|x) + (\theta - 1)r_w(x'|x) + r_i(x'|x) \right) \bigg| x \right] = 1. \quad (11)$$

Substituting the return expressions (6) and (10) into this equations and replacing the log price-dividend ratio $z_i(x) = p_i(x) - d_i(x)$ by its approximation $\hat{z}_i(x; \alpha_i) = \sum_{k=0}^n \alpha_{ik} \Lambda_k(x)$ leads to the residual function

$$\hat{F}_i(x; \alpha_i) = \int_X \left[ \exp \left( \theta \log \delta - \frac{\theta}{\psi} \Delta c(x'|x) + (\theta - 1)\hat{r}_w(x'|x; \alpha_w) + \log \left( e^{\hat{z}_i(x'; \alpha_i)} + 1 \right) - \hat{z}_i(x; \alpha_i) + \Delta d_i(x'|x) \right) - 1 \right] df_x. \quad (12)$$

Recall that the coefficients $\alpha_w$ and thus the function $\hat{r}_w(x'|x; \alpha_w)$ have been computed previously. Therefore, we can now apply one of the projection conditions to solve for the unknown vector $\alpha_i$.

In sum, we apply the projection method twice. In the first step, we approximate the log wealth-consumption ratio $\hat{\gamma}_w(x; \alpha_w)$ by applying the projections on the residual function of the wealth-Euler equation (9). Once $\alpha_w$ is known, the projections can be applied to equation (12)
to solve for the price-dividend ratio \( \hat{z}_i(x; \alpha_i) \) of any asset \( i \). Formally, the algorithm can be described as follows.

**Algorithm** Solving Asset-Pricing Models with Recursive Preferences.

**Initialization.** Define the state space \( X \subset \mathbb{R}^l \); choose the functional forms for \( \hat{z}_w(x; \alpha_w) \) and \( \hat{z}_i(x; \alpha_i) \) as well as the projection method.

**Step 1.** Use the wealth-Euler equation (3) together with the approximated log wealth-consumption ratio \( \hat{z}_w(x; \alpha_w) \) and the definition of the return equation (6) to derive the residual function for the return on wealth

\[
\hat{F}_w(x; \alpha_w) = \int_X \left[ \exp \left( \theta \left( \log \delta + (1 - \frac{1}{\psi}) \Delta c(x'|x) + \hat{z}_w(x') - \log \left( e^{\hat{z}_w(x)} - 1 \right) \right) \right) - 1 \right] df_x.
\]

Compute the unknown solution coefficients \( \alpha_w \) by imposing the projections on \( \hat{F}_w(x; \alpha_w) \).

**Step 2.** Use the solution for the wealth-consumption ratio \( \hat{z}_w(x; \alpha_w) \) and the Euler equation (2) for asset \( i \) together with the approximated log price-dividend ratio \( \hat{z}_i(x; \alpha_i) \) and the definition of the return equation (10) to derive the residual function for asset \( i \),

\[
\hat{F}_i(x; \alpha_i) = \int_X \left[ \exp \left( \theta \log \delta - \frac{\theta}{\psi} \Delta c(x'|x) + (\theta - 1)\hat{r}_w(x'|x; \alpha_w) \right.ight.
\]
\[
+ \log \left( e^{\hat{z}_i(x'; \alpha_i)} + 1 \right) - \hat{z}_i(x; \alpha_i) + \Delta d_i(x'|x) \left.) \right) - 1 \right] df_x.
\]

Compute the unknown solution coefficients \( \alpha_i \) by imposing the projections on \( \hat{F}_i(x; \alpha_i) \).

**Evaluation.** Choose a set of evaluation nodes \( X^e = \{ x_j^e : 1 \leq j \leq m^e \} \subset X \) and compute approximation errors in the residual function of the wealth portfolio and the residual function of asset \( i \). If the errors do not satisfy a predefined error bound, start over at Initialization and change the number of approximation nodes or the degree of the basis functions.

Before we can perform the individual steps of this algorithm, we need to specify additional algorithmic details such as the choices for basis functions and the integration technique.
2.2 Algorithmic Ingredients

In the Initialization step, we need to choose a set of basis functions for the polynomial approximation, a projection method and a set of nodes. To simplify the presentation, we describe the necessary choices for a one-dimensional state space approximated over an interval $X = [x_{\text{min}}, x_{\text{max}}]$. We approximate the solution functions $z_w$ and $z_i$ by Chebyshev polynomials (of the first kind), see Judd (1998). We obtain the Chebyshev polynomials via the recursive relationship

$$T_0(\xi) = 1, \quad T_1(\xi) = \xi, \quad T_{k+1}(\xi) = 2\xi T_k(\xi) - T_{k-1}(\xi),$$

with $T_k : [-1, 1] \rightarrow \mathbb{R}$. Since we need to approximate functions on the domain $X$ and the Chebyshev polynomials are defined on the interval $[-1, 1]$, we need to transform the argument for the polynomials. The basis functions for the approximate solutions $\hat{z}_w(x; \alpha_w)$ and $\hat{z}_i(x; \alpha_i)$ are given by

$$\Lambda_k(x) = T_k \left( 2 \left( \frac{x - x_{\text{min}}}{x_{\text{max}} - x_{\text{min}}} \right) - 1 \right),$$

for $k = 0, 1, \ldots, n$.

We apply two different projection methods in this paper, the collocation method and the Galerkin method. Appendix A provides a brief overview of both types of projection methods. The application of a projection method requires a set of nodes, $x = \{x_j : 0 \leq j \leq m\} \subset X$; we choose the $m+1$ zeros of the Chebyshev polynomial $T_{m+1}$. These points are called Chebyshev nodes,

$$\xi_j = \cos \left( \frac{2j + 1}{2m + 2} \pi \right), \quad j = 0, 1, \ldots, m.$$  

Since all Chebyshev nodes are in the interval $[-1, 1]$, we need to transform them to obtain nodes in the state space $X$. This transformation is

$$x_j = x_{\text{min}} + \frac{x_{\text{max}} - x_{\text{min}}}{2} (1 + \xi_j), \quad j = 0, 1, \ldots, m.$$  

For the collocation method, the number of basis functions, $n + 1$, must be identical to the number of approximation nodes, $m + 1$, and so $m = n$. In Step 1 (and Step 2, if applicable), we must solve the projection conditions involving the residual function. The residual functions defined in equations (9) and (12) contain a conditional expectations operator, which also requires numerical calculations. The underlying exogenous processes in the models we consider are normally distributed, and so we apply Gauss-Hermite quadrature to calculate expectations.

The collocation approach leads to a square system of nonlinear equations, see Appendix A, which can be solved with a standard nonlinear equation solver. The Galerkin projection is slightly more complex, and uses integral operators as projection conditions; these in turn can
be accurately approximated by Gauss-Chebyshev quadrature.

For the Evaluation step we use \( m^e \gg m \) equally spaced evaluation nodes in \( X \) to evaluate the errors in the residual function. In particular, for asset \( i \) we compute the root mean squared errors (RMSE) and maximum absolute errors (MAE) in the residual function (12); these errors are

\[
\text{RMSE}_i = \sqrt{\frac{1}{m^e} \sum_{j=1}^{m^e} \hat{F}_i(x_j^e|\alpha_i)^2}, \quad (14)
\]

\[
\text{MAE}_i = \max_{j=1,2,\ldots,m^e} |\hat{F}_i(x_j^e|\alpha_i)|, \quad (15)
\]

respectively, with

\[
x_j^e = x_{\text{min}} + \frac{x_{\text{max}} - x_{\text{min}}}{m^e - 1} (j - 1), \quad j = 1, \ldots, m^e. \quad (16)
\]

### 2.3 Comparison of Three Families of Methods

In the following two sections we compare the performance of the projection method to commonly used approaches in the literature for solving asset-pricing models with recursive preferences. The two most prominent methods are discretization and linearization techniques. Specifically, we focus on the discretization methods by Tauchen (1986) and Tauchen and Hussey (1991)\(^1\) and the log-linearized pricing kernel approach as described in Bansal and Yaron (2004). Appendix B provides a brief description of the alternative solution methods for which we present results in this paper.

A discussion of log-linearization methods requires careful attention to several important differences among some well-known approaches. Standard log-linearization methods as in Judd (1996) or Collard and Juillard (2001) linearize around the deterministic steady state of the model. In a deterministic model, recursive preferences collapse to the case of CRRA preferences and hence the risk aversion has no influence (as there is no risk). But if the risk aversion has significant influence in the stochastic model, linearizing around the deterministic steady state might not be the best choice. Therefore new techniques have been developed that linearize around the risky steady state of the model (see, for example, Juillard (2011), de Groot (2013) or Meyer-Gohde (2014)).\(^2\) Another drawback of the standard log-linearization is that

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\(^1\)Another discretization method available for one-dimensional AR(1) processes (see e.g. Galindev and Lkhagvasuren (2010) or Kopecky and Suen (2010)) is that of Rouwenhorst (1995). Unfortunately, there is, to the best of our knowledge, no generalization of the method to dimensions higher than one.

\(^2\)These authors define the risky steady state as the state where, in absence of shocks in the current period, the agent decides to stay at the current state while expecting shocks in the future and knowing their probability distribution.
the policies are independent of the volatility of the model (see Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012)). But as Bansal and Yaron (2004) point out, stochastic volatility is one of the key features of the long-run risks model and essential for asset-pricing dynamics. Hence a log-linear approximation for asset-pricing models with recursive preferences and stochastic volatility must account for both features, the risk-adjustment of the steady state and the effects of volatility. Bansal and Yaron (2004) present a linearization technique that meets these requirements which, therefore, has been used extensively for solving asset-pricing models with recursive preferences (Bansal, Kiku, and Yaron (2010), Bansal, Kiku, and Yaron (2012a), Bollerslev, Tauchen, and Zhou (2009), Kaltenbrunner and Lochstoer (2010), Kojien, Lustig, Van Nieuwerburgh, and Verdelhan (2010), Bansal and Shaliastovich (2013), Constantinides and Ghosh (2011), Bansal, Kiku, Shaliastovich, and Yaron (2014) or Beeler and Campbell (2012), among others).

For the comparison of the different solution methods we report numerical solutions, error measures, and running times for two well-known asset-pricing models with recursive preferences. The first asset-pricing model is the endowment economy from Tallarini (2000). Log consumption is modeled as AR(1) deviations from a deterministic linear trend. The deviation from trend is the only state variable in the model and so the state space is one-dimensional. As the model is rather simple and does not account for many empirical features found in the data, we rather view it as a technical analysis to understand the strengths and weaknesses of the different solution methods, in particular with regard to changes in the preference and model parameters. (This exercise is similar in spirit to the approach in Collard and Juillard (2001), except that Collard and Juillard (2001) only consider the case of CRRA utility and hence only focus on the standard log-linearization approach around the deterministic steady state as in Judd (1996).) For the second example, we consider the long-run risks model of Bansal and Yaron (2004), which has gathered much attention recently for its ability to match many financial market characteristics. This model has a two-dimensional state space.

For both models, we first compute Euler errors in the pricing equations on the continuous state space for the projection methods and the log-linearization approach. We consider a continuous state space of $\pm n_\sigma$ standard deviations around the mean of the stationary distribution for each state variable. For the computation of Euler errors, we use $m^e = 100 n_\sigma$ equally spaced evaluation nodes (in each dimension) in the interval $[x_{min}, x_{max}]$ to evaluate the (absolute) error in the pricing equation. We use 8 quadrature nodes for both the Gauss-Hermite quadrature to solve the integral in the pricing equation and the Gauss-Chebyshev quadrature for the integral that arises in the Galerkin projection.3 Regarding the efficiency of the projection approach, practical experience has shown, that good initial guesses can lead to

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3We also considered solutions with more quadrature nodes, but the results did not change significantly.
significant reductions of computation times especially for higher order approximations. Hence we take the lower degree solutions as initial guesses for the higher degree approximations. For example to compute a degree-6 approximation we first compute a degree-3 approximation and take that as an initial guess. Computations times are always stated as total times including the computation time of the initial guess. All results are computed in Matlab with the solver ‘fmincon’. (We solve nonlinear systems of equations as optimization problems with a dummy objective function and the nonlinear equations as the constraints of the optimization problem.) We use fmincon’s active-set algorithm with an error tolerance of $10^{-8}$.

In addition to the Euler errors, we also report errors in economically meaningful variables such as the mean and standard deviation of the wealth-consumption and the price-dividend ratio. To do so, we compute the “true” solution by using a very large number of nodes in the Tauchen and Hussey (1991) procedure or using a Chebyshev approximation of extremely high degree. While these calculations might take a very long time (more than two days for the Tauchen and Hussey (1991) procedure!), they allow us to evaluate numerical results for discretizations with fewer nodes, for projections with polynomials of smaller degree, and for the linearization approach.

3 The Endowment Economy of Tallarini (2000)

We consider the endowment economy of Tallarini (2000). Log consumption, $c_t$, is modeled as simple AR(1) deviations from a linear trend,

$$
c_t = \mu t + x_t
$$

$$
x_t = \rho x_{t-1} + \sigma \epsilon_t, \quad \epsilon_t \sim N(0, 1),
$$

where $\mu$ is the average net growth rate of consumption, $\rho$ is the degree of persistence and the state of the economy is described by the one-dimensional process $x_t$. In the following analysis, we focus exclusively on the pricing of the wealth portfolio (Step 1 in our solution algorithm) and make no further assumptions about dividends in the model. For the projection methods we approximate the solution over the range of $\pm n_\sigma$ standard deviations around the mean of the stationary distribution for $x_t$. The first two moments of this distribution are $E(x_t) = 0$ and $\sigma(x_t) = \sigma_t / \sqrt{1 - \rho^2}$.

For the Bansal-Yaron log-linearization approach the log wealth-consumption ratio is a
linear function of the state \( x_t \) of the economy,

\[ z_w(x_t) = A_{0,w} + A_{1,w}x_t. \] (19)

Appendix B describes the derivation of the unknown solution coefficients \( A_{0,w} \) and \( A_{1,w} \) and Appendix C reports analytical expressions for the coefficients.

### 3.1 Approximation Errors in the Euler Equations

Tables 1 and 2 show Euler approximation errors for the pricing of the wealth portfolio for the two projection methods (for degrees 3, 6, and 9), and the Bansal-Yaron log-linearization approach. We report Euler errors for six combinations of the preference parameters, \( \gamma \) and \( \psi \). The first two combinations correspond to CRRA preferences with a low \( (\gamma = 2, \psi = 0.5) \) and a high \( (\gamma = 10, \psi = 0.1) \) degree of risk aversion, respectively. The third combination is the parameter estimates from Bansal and Yaron (2004) \( (\gamma = 10 \text{ and } \psi = 1.5) \). Table 1 reports Euler errors for these three cases. Table 2 depicts errors for three more cases of Epstein-Zin preferences, namely for \( \gamma = 10, \psi = 0.5 \), for \( \gamma = 7.5, \psi = 1.5 \), and for \( \gamma = 2, \psi = 1.5 \).

The leftmost column indicates the size of approximating interval by providing the number \( n_\sigma \) of standard deviations around the mean of the stationary distribution of the model. The parameters for the consumption process are taken from Pohl, Schmedders, and Wilms (2014) and are given by \( \sigma_\epsilon = 0.0343, \mu = 0.02, \rho = 0.91 \). We set \( \delta = 0.99 \). We later vary these parameters to analyze their influence on the performance of the different solution methods.

We observe that the Euler errors for the Bansal-Yaron log-linearization approach depend strongly on the evaluation range, \( n_\sigma \), and the preference parameters, \( \gamma \) and \( \psi \). For the standard case of CRRA utility with \( \gamma = 2 \) and \( \psi = 0.5 \) the maximum absolute approximation error is as small as 0.0076 even for 10 standard deviations around the stationary mean \( (n_\sigma = 10) \). The Euler errors increase dramatically for large values of risk aversion \( (\gamma = 10 \text{ and } \psi = 0.1) \) with the maximum error being as large as 289 for \( n_\sigma = 10 \). For the parameter set of Bansal and Yaron (2004), \( \gamma = 10, \psi = 1.5 \), the approximation errors become significantly smaller, so the log-linearization appears to provide a good approximation of the model.

For the projection methods, already for the degree-6 approximations the Euler errors are several orders of magnitude smaller than those of the log-linearization approach. Moreover, increasing the approximation degree leads to highly accurate solutions even for the larger approximation interval \( n_\sigma = 10 \). Both projection methods are also very robust to changes in the preference parameters. These findings are confirmed for the second parameter set.

---

Table 1: Euler Approximation Errors for the Bansal-Yaron Log-Linearization and Projection Methods in the Model of Tallarini (2000)

<table>
<thead>
<tr>
<th></th>
<th>Log-Lin</th>
<th>Collocation</th>
<th>Galerkin</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n = 3$</td>
<td>$n = 6$</td>
<td>$n = 9$</td>
</tr>
<tr>
<td></td>
<td>$n = 3$</td>
<td>$n = 6$</td>
<td>$n = 9$</td>
</tr>
<tr>
<td>$n_{\sigma} = 3$</td>
<td>4.8e-4</td>
<td>5.1e-6</td>
<td>3.0e-10</td>
</tr>
<tr>
<td></td>
<td>2.4e-4</td>
<td>3.5e-6</td>
<td>1.9e-10</td>
</tr>
<tr>
<td></td>
<td>0.0076</td>
<td>1.9e-4</td>
<td>6.3e-8</td>
</tr>
<tr>
<td></td>
<td>0.0028</td>
<td>1.2e-4</td>
<td>3.8e-8</td>
</tr>
<tr>
<td>$n_{\sigma} = 10$</td>
<td>0.0912</td>
<td>0.0024</td>
<td>1.9e-6</td>
</tr>
<tr>
<td></td>
<td>0.0305</td>
<td>8.7e-7</td>
<td>7.4e-9</td>
</tr>
<tr>
<td></td>
<td>289</td>
<td>0.1266</td>
<td>0.0018</td>
</tr>
<tr>
<td></td>
<td>12.52</td>
<td>0.0451</td>
<td>0.0011</td>
</tr>
</tbody>
</table>

| $n_{\sigma} = 3$| 0.0124  | 1.9e-6      | 1.1e-8   |
|                 | 0.0011  | 8.7e-7      | 7.4e-9   |
| $n_{\sigma} = 10$| 0.0035  | 7.3e-5      | 6.3e-10  |
|                 | 7.3e-5  | 6.4e-10     | 9.4e-11  |

The table shows Euler approximation errors for the pricing of the wealth return in the model of Tallarini (2000) for the Bansal-Yaron log-linearization and the projections methods for different degrees of polynomial approximation $n$. Errors are reported for different sets of preference parameters $\gamma$ and $\psi$. The first row for each pair of parameters shows the MAE and the second row the RMSE at $n_e = 100n_{\sigma}$ uniformly distributed evaluation nodes within $n_{\sigma}$ standard deviations around the mean of the stationary distribution of the model.
The table shows Euler approximation errors for the pricing of the wealth return in the model of Tallarini (2000) for the Bansal-Yaron log-linearization and the projections methods for different degrees of polynomial approximation $n$. Errors are reported for different sets of preference parameters $\gamma$ and $\psi$. The first row for each pair of parameters shows the MAE and the second row the RMSE at $m_\epsilon = 100n_\sigma$ uniformly distributed evaluation nodes within $n_\sigma$ standard deviations around the mean of the stationary distribution of the model.
(see Table 2). Also, both the collocation and the Galerkin method produce about the same magnitude of errors for the same degree \( n \) of the polynomial approximation.

### 3.2 Approximation Errors in the Wealth-Consumption Ratio

Next we analyze the implications of errors in the Euler equation on quantities of economic interest. For this purpose we report the relative errors (in absolute value) in the unconditional mean and the standard deviation of the log wealth-consumption ratio for the different methods. In Table 3 (first set of preference parameters) and Table 4 (second set of preference parameters) we report relative errors as well as the computation times for different sets of preference parameters and approximation degrees.

We observe that the projection methods show very low approximation errors already for the degree-3 approximations and that the results are very robust across the different preference parameters. Additionally, the degree-6 solutions provide about the same accuracy as the degree-9 approximations, which suggests that already the degree-6 approximations are able to capture most of the non-linearities in the fixed approximation interval. Note, that the degree-9 approximations might further decrease approximation errors if we increased the width of the approximation range \( 2n\sigma \). The Bansal-Yaron log-linearization produces relatively small approximation errors for all sets of parameters except for \( \gamma = 10, \psi = 0.1 \) where the errors are slightly larger.

The approximation errors of the discretization techniques are rather small for the mean of the wealth-consumption ratio but for the standard deviation the methods show difficulties and a larger number of nodes is needed to obtain accurate solutions. Comparing the different discretization methods we find that the adjustment by Floden (2007) (TH-F) slightly improves on the original method (TH) as proposed in Tauchen and Hussey (1991), particularly in the speed of convergence as the number of discretization nodes increases. Tauchen (1986)’s method often exhibits even smaller errors for the 3-node discretization but does not converge as fast as TH-F. Compared to the projection methods, the discretizations perform significantly worse while requiring about the same computation time. Log-linearization is the fastest method (usually about twice as fast as the degree-3 polynomial approximations) in this example, but it does not achieve the same accuracy as the projection methods.

We briefly summarize the performance of the three families of numerical solution methods for the endowment economy of Tallarini (2000). The two projection methods deliver Euler approximation errors as well as relative errors for the first two moments of the wealth-consumption ratio that are typically several orders of magnitude smaller than the corresponding values for the discretization and log-linearization methods. However, this considerable
Table 3: Absolute Relative Errors in the Unconditional Mean and Standard Deviation of the Log Wealth-Consumption Ratio in the Model of Tallarini (2000)

<table>
<thead>
<tr>
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<th>Collocation</th>
<th>Galerkin</th>
<th>BY Log-Lin</th>
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</thead>
<tbody>
<tr>
<td>$E (w_t - c_t)$</td>
<td>5.3e-7</td>
<td>4.0e-10</td>
<td>3.5e-10</td>
</tr>
<tr>
<td>$\sigma (w_t - c_t)$</td>
<td>3.7e-4</td>
<td>2.9e-10</td>
<td>2.8e-10</td>
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<tr>
<td>Time</td>
<td>0.0217</td>
<td>0.0375</td>
<td>0.0549</td>
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<table>
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<th>Tauchen</th>
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<td>1.3e-5</td>
</tr>
<tr>
<td>$\sigma (w_t - c_t)$</td>
<td>0.3390</td>
<td>0.2649</td>
<td>0.0059</td>
</tr>
<tr>
<td>Time</td>
<td>0.0248</td>
<td>0.0277</td>
<td>0.0413</td>
</tr>
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<th>Collocation</th>
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<th>BY Log-Lin</th>
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</thead>
<tbody>
<tr>
<td>$E (w_t - c_t)$</td>
<td>7.2e-4</td>
<td>4.5e-9</td>
<td>4.5e-9</td>
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<tr>
<td>$\sigma (w_t - c_t)$</td>
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<td>1.5e-7</td>
<td>1.5e-7</td>
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<tr>
<td>Time</td>
<td>0.0242</td>
<td>0.0236</td>
<td>0.0525</td>
</tr>
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<th>TH-F</th>
<th>Tauchen</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E (w_t - c_t)$</td>
<td>0.0418</td>
<td>0.0361</td>
<td>0.0010</td>
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<tr>
<td>$\sigma (w_t - c_t)$</td>
<td>0.0979</td>
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<td>0.0016</td>
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<tr>
<td>Time</td>
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<td>0.0252</td>
<td>0.0376</td>
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</tr>
</thead>
<tbody>
<tr>
<td>$E (w_t - c_t)$</td>
<td>8.7e-8</td>
<td>1.3e-11</td>
<td>1.2e-11</td>
</tr>
<tr>
<td>$\sigma (w_t - c_t)$</td>
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<td>1.0e-11</td>
<td>5.0e-12</td>
</tr>
<tr>
<td>Time</td>
<td>0.0224</td>
<td>0.0217</td>
<td>0.0494</td>
</tr>
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<table>
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<th>TH-F</th>
<th>Tauchen</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E (w_t - c_t)$</td>
<td>6.3e-5</td>
<td>5.3e-6</td>
<td>1.7e-6</td>
</tr>
<tr>
<td>$\sigma (w_t - c_t)$</td>
<td>0.4234</td>
<td>0.2403</td>
<td>0.0074</td>
</tr>
<tr>
<td>Time</td>
<td>0.0277</td>
<td>0.0237</td>
<td>0.0350</td>
</tr>
</tbody>
</table>

The table shows absolute relative errors in the unconditional mean and standard deviation of the log wealth-consumption ratio as well as the computation times for different sets of preference parameters. For the projection methods we set $n_{c} = 3$ and for the Tauchen (1986) method $m_{T} = 2$. 

\[\gamma = 2, \psi = 0.5\]

\[\gamma = 10, \psi = 0.1\]

\[\gamma = 10, \psi = 1.5\]
Table 4: Absolute Relative Errors in the Unconditional Mean and Standard Deviation of the Log Wealth-Consumption Ratio in the Model of Tallarini (2000) - Second Parameter Set

<table>
<thead>
<tr>
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<th>Collocation</th>
<th>Galerkin</th>
<th>BY Log-Lin</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>$E(w_t - c_t)$</td>
<td>2.2e-7</td>
<td>2.2e-7</td>
<td>2.2e-7</td>
</tr>
<tr>
<td>$\sigma(w_t - c_t)$</td>
<td>1.5-5</td>
<td>1.5-5</td>
<td>1.5e-5</td>
</tr>
<tr>
<td>Time</td>
<td>0.0225</td>
<td>0.0304</td>
<td>0.0393</td>
</tr>
</tbody>
</table>

|                        | TH |         |         |         |         |         |         |
|                        | 3 | 6 | 9 | 3 | 6 | 9 |          |
| $E(w_t - c_t)$         | 3.0e-5 | 1.4-5 | 6.6e-6 | 2.2e-5 | 4.0e-6 | 8.5e-7 | 3.5e-5 | 5.3e-6 | 1.3e-6 |
| $\sigma(w_t - c_t)$    | 0.4235 | 0.1902 | 0.0862 | 0.2403 | 0.0423 | 0.0074 | 0.1475 | 0.0382 | 0.0284 |
| Time                   | 0.0246 | 0.0295 | 0.0380 | 0.0203 | 0.0255 | 0.0325 | 0.0222 | 0.0274 | 0.0361 |

|                        | TH |         |         |         |         |         |         |
|                        | 3 | 6 | 9 | 3 | 6 | 9 |          |
| $E(w_t - c_t)$         | 6.4e-6 | 4.8e-9 | 4.8e-9 | 6.4e-6 | 2.8e-9 | 2.6e-9 | 6.0e-4 |
| $\sigma(w_t - c_t)$    | 3.5e-4 | 8.6e-9 | 8.6e-9 | 3.5e-4 | 1.6e-9 | 9.0e-10 | 0.0021 |
| Time                   | 0.0234 | 0.0335 | 0.0419 | 0.0236 | 0.0388 | 0.0595 | 0.0150 |

|                        | TH |         |         |         |         |         |         |
|                        | 3 | 6 | 9 | 3 | 6 | 9 |          |
| $E(w_t - c_t)$         | 0.0029 | 0.0017 | 8.6e-4 | 0.0015 | 3.5e-4 | 7.7e-5 | 0.0033 | 2.3e-5 | 4.7e-4 |
| $\sigma(w_t - c_t)$    | 0.3392 | 0.1320 | 0.0550 | 0.2625 | 0.0384 | 0.0054 | 0.2203 | 0.0722 | 0.0018 |
| Time                   | 0.0261 | 0.0311 | 0.0395 | 0.0216 | 0.0259 | 0.0326 | 0.0244 | 0.0285 | 0.0360 |

|                        | TH |         |         |         |         |         |         |
|                        | 3 | 6 | 9 | 3 | 6 | 9 |          |
| $E(w_t - c_t)$         | 5.7e-7 | 1.6e-11 | 1.6e-11 | 5.7e-7 | 8.7e-11 | 8.6e-12 | 6.2e-5 |
| $\sigma(w_t - c_t)$    | 1.5e-5 | 1.0e-11 | 1.0e-11 | 1.5e-5 | 1.0e-11 | 5.0e-12 | 1.7e-5 |
| Time                   | 0.0229 | 0.0336 | 0.0410 | 0.0225 | 0.0386 | 0.0567 | 0.0135 |

|                        | TH |         |         |         |         |         |         |
|                        | 3 | 6 | 9 | 3 | 6 | 9 |          |
| $E(w_t - c_t)$         | 3.4e-5 | 2.6e-5 | 1.5e-5 | 2.3e-6 | 2.9e-6 | 5.4e-7 | 3.43e-4 | 9.6e-6 | 1.2e-5 |
| $\sigma(w_t - c_t)$    | 0.4235 | 0.1902 | 0.0862 | 0.2403 | 0.0423 | 0.0074 | 0.1475 | 0.0382 | 0.0284 |
| Time                   | 0.0248 | 0.0295 | 0.0347 | 0.0208 | 0.0260 | 0.0305 | 0.0236 | 0.0291 | 0.0356 |

The table shows absolute relative errors in the unconditional mean and standard deviation of the wealth-consumption ratio as well as the computation times for different sets of preference parameters. For the projection methods we set $n_\sigma = 3$ and for the Tauchen (1986) method $m_T = 2$.  

18
outperformance does not appear to be relevant from the viewpoint of economics. The log-linearization method has the largest relative errors for the parametrization $\gamma = 10, \psi = 0.1$, namely 3.61% for the average wealth-consumption ratio and 7.75% for its standard deviation. (The largest errors for the 9-node Tauchen procedure are smaller.) However, while errors of this size may be annoying, they hardly matter for a qualitative interpretation of an economic model. In fact, errors of this magnitude may not even matter in a quantitative economic analysis. To put it bluntly, the great numerical advantage of the projection methods over the other two family of methods does not bear relevance to the economic analysis of the model. In Section 4 we demonstrate that such a conclusion in favor of the simpler but inferior solution methods is not correct in general, particularly for more complex models.

In the next step of the analysis, we evaluate the robustness of the methods with regard to changes in the underlying parameter estimates. The exercise serves to analyze the factors that drive the accuracy of the different approximation methods and to understand, why and in which cases the methods fail to compute accurate solutions. In particular it delivers interesting insights regarding suitable solution methods for the long-run risks model in Section 4. Therefore we focus on the preference parameters $\gamma = 10$ and $\psi = 1.5$ which are the parameters used in the long-run risks model of Bansal and Yaron (2004).

### 3.3 Robustness with Regard to Changes in the Input Parameters

In this subsection we evaluate the performance of the solution methods with regard to their sensitivity to changes in the model parameters. Figure 1 shows the approximation errors as in Table 3 for different values of $\rho$ and Figure 2 shows the corresponding errors for variations in $\sigma_e$. We observe that the performances of the Bansal-Yaron log-linearization and the discretization methods depend on the model parameters. The approximation error of the Bansal-Yaron log-linearization increases strongly with the serial correlation of the underlying process and also with the volatility. (Note the log10 scale.) This result is especially interesting, as the long-run risks model, considered in the next section, relies on highly persistent processes. The approximation errors of Tauchen and Hussey (1991) also increase with the serial correlation $\rho$. This is a well documented fact in the literature (see Floden (2007)). The method of Floden (2007) performs better, but also shows severe difficulties in approximating second-order moments. Tauchen’s method is more robust with regard to changes in $\rho$. The errors for the projection methods on the other hand are difficult to distinguish from zero with maximum errors in the order of $10^{-10}$ and prove to be very robust with regard to changes in both parameters.

The results show that while all the methods can provide more or less accurate solutions for
The graph shows absolute relative errors in the unconditional mean (left panel) and standard deviation (right panel) of the wealth-consumption ratio for different values of $\rho$ in log10 scale. For the projection methods a degree-9 approximation is used with $n_{\sigma} = 3$. For the discretizations 9 nodes are used and $m_T = 2$. 
The graph shows absolute relative errors in the unconditional mean (left panel) and standard deviation (right panel) of the wealth-consumption ratio for different values of $\sigma_\epsilon$ in log10 scale. For the projection methods a degree-9 approximation is used with $n_{\sigma} = 3$. For the discretizations 9 nodes are used and $m_T = 2$. 

Figure 2: Absolute Relative Errors in the Pricing of the Mean and Standard Deviation of the Log Wealth-Consumption Ratio as a Function of $\sigma_\epsilon$ in the Model of Tallarini (2000)
the simple endowment economy considered in this section, the accuracy of the Bansal-Yaron log-linearization and the discretization methods depends strongly on the parametrization of the underlying process. In particular, we find that high serial correlation or large volatilities can significantly increase approximation errors.

In the following section, we analyze the performance of the different methods for solving the long-run risks model of Bansal and Yaron (2004). This model features highly persistent processes not only for the long run growth component but also for the stochastic volatility. Therefore, in light of our results in this subsection, we may expect that methods such as the Bansal-Yaron log-linearization or the discretization approach might induce large approximation errors for models with such features.

4 The Long-Run Risks Model of Bansal and Yaron (2004)

As the second application we consider the long-run risks model of Bansal and Yaron (2004). The main innovation in the model is that growth rates feature random but highly persistent long-run shocks. Additionally the conditional variance of the growth rates is itself stochastic. So the model has two state variables, the long-run component, \( x_t \), and the variance level, \( \sigma_t^2 \). The model has emerged as a workhorse asset-pricing model, and there have been many variations and extensions that use the same log-linear approximation as in Bansal and Yaron (2004). For example Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010) and Bansal and Shaliastovich (2013) add a third highly persistent process to model inflation, Bansal, Kiku, and Yaron (2010) adds a third state variable to model business cycle risk, and Kaltenbrunner and Lochstoer (2010) analyzes how long-run risk arises in a production economy. In this paper we focus on the original model from Bansal and Yaron (2004) and show that even the standard model has highly nonlinear policy functions that can introduce large approximation errors depending on the calibration of the model. Hence when using log-linear approximations for further extensions of the model, particular attention should be paid to the accuracy of the solution. Bansal and Yaron (2004) specify the original model as follows.

\[
\Delta c_{t+1} = \mu_c + x_t + \sigma_t \eta_{t+1} \\
x_{t+1} = \rho x_t + \phi \sigma_t \epsilon_{t+1} \\
\sigma_{t+1}^2 = \bar{\sigma}^2 (1 - \nu) + \nu \sigma_t^2 + \sigma_\omega \omega_{t+1} \\
\Delta d_{t+1} = \mu_d + \Phi x_t + \phi \sigma_t \omega_{t+1} + \pi \sigma_t \eta_{t+1} \\
\eta_{t+1}, \epsilon_{t+1}, \omega_{t+1}, u_{t+1} \sim i.i.d. N(0, 1).
\]
We consider two sets of parameter values for the model. The original set used in Bansal and Yaron (2004) and the more recent set in Bansal, Kiku, and Yaron (2012a). The parameters are calibrated to match annual financial market characteristics for the period from 1930–2008 while the representative agent has a monthly decision interval. Table 5 lists the two sets of parameter estimates. The main difference between the two sets of parameter values is that in the new calibration of Bansal, Kiku, and Yaron (2012a), the persistence of the volatility shock is higher and that shocks to dividends are correlated with short-run shocks to consumption growth ($\pi = 2.6$ in the new calibration compared to $\pi = 0$ in the original calibration). These changes increase the influence of the volatility channel compared to the long-run risks channel of the model. The adjustment is needed to get rid of some implications of the original calibration that are inconsistent with the data. In particular, as for example Zhou and Zhu (2014) or Beeler and Campbell (2012) point out for the original 2004 calibration, the log price-dividend ratio has predictive power for future consumption growth, while this relationship is not present in the data. By increasing the influence of the volatility channel, this predictability vanishes. Therefore, in this paper we focus particularly on the Bansal, Kiku, and Yaron (2012a) calibration, as it displays more consistency with key statistics of financial markets data (than the original 2004 calibration).

We solve the model for the return of the wealth portfolio, $z_w$, the market portfolio, $z_m$, and the risk-free rate, $z_{rf}$. As in the previous section, we first examine Euler errors for the continuous methods and evaluate all methods with respect to their ability to compute unconditional moments of the model variables. Afterwards we conduct a full evaluation of the different methods with respect to their ability to approximate different economically relevant moments on an annual basis. To compute the annualized moments, we simulate 1,000,000 years of artificial data. Beeler and Campbell (2012) provide a detailed description of how to compute the annual moments from the monthly observations. A significant issue in the model is that the variance process $\sigma_t^2$ can, in fact, become negative. To overcome this problem, Bansal and Yaron (2004) replace all negative realizations with very small but positive values. We proceed in the same way for all methods to achieve consistent results. For the approximation interval

<table>
<thead>
<tr>
<th></th>
<th>$\mu_c$</th>
<th>$\rho$</th>
<th>$\phi_e$</th>
<th>$\bar{\sigma}$</th>
<th>$\nu$</th>
<th>$\sigma_\omega$</th>
<th>$\mu_d$</th>
<th>$\Phi$</th>
<th>$\phi$</th>
<th>$\pi$</th>
<th>$\gamma$</th>
<th>$\psi$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BKY (2012a)</td>
<td>1.5e-3</td>
<td>0.975</td>
<td>0.038</td>
<td>7.2e-3</td>
<td>0.999</td>
<td>2.8e-6</td>
<td>1.5e-3</td>
<td>2.5</td>
<td>5.96</td>
<td>2.6</td>
<td>10</td>
<td>1.5</td>
<td>0.9989</td>
</tr>
<tr>
<td>BY (2004)</td>
<td>1.5e-3</td>
<td>0.979</td>
<td>0.044</td>
<td>7.8e-3</td>
<td>0.987</td>
<td>2.3e-6</td>
<td>1.5e-3</td>
<td>3.0</td>
<td>4.5</td>
<td>0</td>
<td>10</td>
<td>1.5</td>
<td>0.998</td>
</tr>
</tbody>
</table>

*The table shows the parameters estimates of the long-run risks model as reported in Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012a).*
of the projection methods we choose the interval to be slightly larger than the maximum observation range of the long simulations. The values are given by $x_{\text{min}} = -0.013$, $x_{\text{max}} = 0.013$, $\sigma^2_{\text{min}} = 0$ and $\sigma^2_{\text{max}} = 0.00038$. For the collocation method we use the full tensor product of one-dimensional basis functions, which allows us to use Chebyshev nodes in each dimension and still maintain an exactly identified system of equations—that is, $(n+1)^2$ unknown solution coefficients and $(m+1)^2$ approximation nodes with $n = m$. For the Galerkin method we choose instead the set of complete polynomials, which are the products of one-variable polynomials such that the total degree is at most $n+1$. This choice reduces the number of unknown solution coefficients from $(n+1)^2$ to $(n+1)^2/2 + (n+1)/2$ and thus lowers the computational costs without much loss of approximation quality.

For the Bansal-Yaron log-linearization approach, the log wealth-consumption ratio $z_w$, the log price-dividend ratio $z_m$, and the log risk-free rate are linear functions of the state variables,

$$z_w(x_t, \sigma^2_t) = A_{0,w} + A_{1,w}x_t + A_{2,w}\sigma^2_t$$
$$z_m(x_t, \sigma^2_t) = A_{0,m} + A_{1,m}x_t + A_{2,m}\sigma^2_t$$
$$z_{rf}(x_t, \sigma^2_t) = A_{0,rf} + A_{1,rf}x_t + A_{2,rf}\sigma^2_t.$$

Appendix B describes the derivation of the unknown solution coefficients and Appendix C reports analytical expressions for all nine coefficients.

### 4.1 Approximation Errors in the Euler Equations

Table 6 shows Euler errors for the wealth, market, and risk-free return. The first row of each entry shows the maximum absolute error and the second row the root mean squared error. We observe that already the degree-3 polynomial approximation delivers errors that are usually about 2 orders of magnitude smaller than those of the linear approximation. The collocation performs slightly better than Galerkin projection in most cases, which might be driven by the fact that we use complete polynomials for the Galerkin projection and tensor products for the collocation. As we document below, using complete polynomials can lead to significant gains in computation time. With a few exceptions, we observe that approximation errors are larger for the parameter values in Bansal, Kiku, and Yaron (2012a) than for the corresponding values in Bansal and Yaron (2004). In the subsequent sections we analyze the observed phenomena and their causes.

---

6We also tried larger intervals which didn’t significantly change the results.
### Table 6: Euler Approximation Errors for the Bansal-Yaron Log-Linearization and Projection Methods in the Long-Run Risks Model

<table>
<thead>
<tr>
<th>BY Log-Lin</th>
<th>Collocation</th>
<th>Galerkin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibration BKY (2012a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$n = 3$</td>
<td>$n = 6$</td>
</tr>
<tr>
<td>Wealth-Euler</td>
<td>0.0160</td>
<td>3.9e-5</td>
</tr>
<tr>
<td></td>
<td>0.0044</td>
<td>1.6e-5</td>
</tr>
<tr>
<td>Market-Euler</td>
<td>0.0144</td>
<td>1.9e-4</td>
</tr>
<tr>
<td></td>
<td>0.0022</td>
<td>6.6e-5</td>
</tr>
<tr>
<td>Risk-Free-Euler</td>
<td>0.0168</td>
<td>4.1e-5</td>
</tr>
<tr>
<td></td>
<td>0.0046</td>
<td>1.7e-5</td>
</tr>
<tr>
<td>Calibration BY (2004)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$n = 3$</td>
<td>$n = 6$</td>
</tr>
<tr>
<td>Wealth-Euler</td>
<td>0.0018</td>
<td>1.2e-5</td>
</tr>
<tr>
<td></td>
<td>4.3e-4</td>
<td>6.3e-6</td>
</tr>
<tr>
<td>Market-Euler</td>
<td>0.0119</td>
<td>2.3e-4</td>
</tr>
<tr>
<td></td>
<td>0.0023</td>
<td>1.3e-4</td>
</tr>
<tr>
<td>Risk-Free-Euler</td>
<td>0.0019</td>
<td>1.2e-5</td>
</tr>
<tr>
<td></td>
<td>4.5e-4</td>
<td>6.6e-6</td>
</tr>
</tbody>
</table>

The table shows Euler approximation errors for the pricing of the wealth return, the market return and the risk-free rate for the Bansal-Yaron log-linearization and the projections methods for different degrees of polynomial approximation $n$. Errors are reported for the parameter specifications in Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012a), respectively. The first row for each Euler equation shows the maximum absolute error and the second row the root mean squared error of 200 uniformly distributed evaluation nodes in each dimension.
4.2 Approximation Errors in the Wealth-Consumption and Price-Dividend Ratio

Table 7 shows absolute relative errors in the unconditional mean and standard deviation of the monthly log wealth-consumption and log price-dividend ratio, as well as computation times. We find that the linearization does a reasonably good job for the parameters in Bansal and Yaron (2004) with a maximum error of 1.53% in the volatility of the price-dividend ratio. For the parameter set of Bansal, Kiku, and Yaron (2012a) the results are considerably worse. The linearization has particular difficulties in approximating second order moments, with a maximum error of 26.9% in the volatility of the price-dividend ratio. We document below that these large errors are driven by deviations from linearity in the model’s solution. In particular, the linearization approach assumes that first derivatives of the solution are approximately constant and the second derivatives are approximately zero. In Section 4.3 we show numerically that this assumption fails to hold; this failure leads to large approximation errors, particularly with respect to the volatility of the asset returns. On the contrary, the projection methods show much smaller approximation errors already for the degree-3 approximation. As in the previous section for the model with a one-dimensional state space, we find that the degree-6 solutions provide about the same accuracy as the degree-9 approximations; this finding suggests that already the degree-6 approximations are able to capture most of the non-linearities in the fixed approximation interval. Tauchen and Hussey (1991)’s method is not able to produce reliable results using a 10-node discretization and shows very slow convergence properties, particularly for the calibration by Bansal, Kiku, and Yaron (2012a) with its large value for the persistence parameter $\nu$. Again we find that all discretization methods show particular difficulties in approximating the second order dynamics. The computation time of the discretization methods increases dramatically with the number of discretization nodes. For example, for 10 nodes the Tauchen and Hussey (1991) method takes about 3 seconds to compute the optimal solution, while it takes about 50 minutes to compute the 50-node approximation. The projection methods take less than two seconds to compute the degree-3 and less than five seconds for the degree-6 approximations, which already provide highly accurate results. In addition the Galerkin method is a bit faster than the collocation approach.

4.3 Understanding the Non-Linearities of the Long-Run Risk Model

Figures 3 (calibration BKY (2012)) and 4 (calibration BY (2004)) show isolines for the absolute errors in the log wealth-consumption ratio (left panel) and the log price dividend ratio (right panel) of the Bansal-Yaron log-linearization as a function of the states $x$ and $\sigma^2$ (black solid
Table 7: Absolute Relative Errors in the Unconditional Mean and Standard Deviation of the Log Wealth-Consumption and Log Price-Dividend Ratio in the Long-Run Risks Model

<table>
<thead>
<tr>
<th></th>
<th>Collocation</th>
<th>Galerkin</th>
<th>BY Log-Lin</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>$E(w_t - c_t)$</td>
<td>1.5e-5</td>
<td>2.0e-8</td>
<td>2.0e-8</td>
</tr>
<tr>
<td>$\sigma(w_t - c_t)$</td>
<td>4.6e-4</td>
<td>4.9e-7</td>
<td>4.9e-7</td>
</tr>
<tr>
<td>$E(p_t - d_t)$</td>
<td>0.0024</td>
<td>2.9e-8</td>
<td>2.9e-8</td>
</tr>
<tr>
<td>$\sigma(p_t - d_t)$</td>
<td>0.0035</td>
<td>9.4e-6</td>
<td>9.4e-6</td>
</tr>
<tr>
<td>Time</td>
<td>1.01</td>
<td>4.76</td>
<td>12.14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>TH</th>
<th>TH-F</th>
<th>Tauchen</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>30</td>
<td>50</td>
</tr>
<tr>
<td>$E(w_t - c_t)$</td>
<td>0.0794</td>
<td>0.0608</td>
<td>0.0533</td>
</tr>
<tr>
<td>$\sigma(w_t - c_t)$</td>
<td>0.9521</td>
<td>0.8627</td>
<td>0.8240</td>
</tr>
<tr>
<td>$E(p_t - d_t)$</td>
<td>0.2533</td>
<td>0.1223</td>
<td>0.0877</td>
</tr>
<tr>
<td>$\sigma(p_t - d_t)$</td>
<td>0.8881</td>
<td>0.6981</td>
<td>0.6267</td>
</tr>
<tr>
<td>Time</td>
<td>2.35</td>
<td>228.8</td>
<td>2944</td>
</tr>
</tbody>
</table>

The table shows absolute relative errors in the unconditional mean and standard deviation of the monthly log wealth-consumption and log price-dividend ratio as well as the computation times. Errors are reported for the parameter specifications in Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012a), respectively.
For example along a line marked with ‘0.1’, the absolute error of the Bansal-Yaron log-linearization is 0.1. The figures also show the density of $10^7$ simulated observations of the states $x$ and $\sigma^2$. The density demonstrates where the model ‘lives’ most of the time, but it also shows the regions that are reached in the long simulations and therefore also might be important for model outcomes. Corresponding errors for the first derivatives with respect to the state variables are shown in Figures 5 and 6 and for the second derivatives in Figures 7 and 8.

We find that for the calibration of BKY (2012) the errors in the log wealth-consumption are rather small with maximum values of about 0.16 within the observation range. For the log price-dividend ratio, the errors are also small in the area close to the long run mean of the processes, but they increase significantly with $\sigma^2$ and reach values of up to 0.75 within the observation range, see Figure 3. Or put differently, the price dividend ratio obtained by the Bansal-Yaron log-linearization is off by factors larger than 2. For the BY (2004) calibration the errors are significantly smaller and don’t show the strict tendency to increase with $\sigma^2$, see Figure 4.

The errors in the first derivatives show similar patterns. Again the errors in the derivatives of the price-dividend ratio are significantly larger than the errors in the derivatives of the wealth-consumption ratio and the errors strictly increase with $\sigma^2$ for the BKY (2012) calibration. We observe in Figure 5 that the errors in the derivatives with respect to $\sigma^2$ are especially large with errors up to 3000 for the price-dividend ratio, while they are considerably smaller for the BY (2004) calibration with all errors being smaller than 650, see Figure 6. As mentioned above, the main purpose of the BKY (2012) calibration is to amplify the role of the stochastic volatility channel by increasing its persistence. But as demonstrated in the figures, this effect introduces large non-linearities to the model that cannot be captured by the Bansal-Yaron log-linearization and hence causes large approximation errors.

Finally, Figures 7 and 8 show that the second derivatives in the model are substantially different from 0 (which is the value assumed by the Bansal-Yaron log-linearization) and they are especially large (more than $10^5$!) for the second derivative with respect to $\sigma^2$. This missing curvature does not have a large impact on the first-order moments of the model, but does have a large impact on the second-order moments, as reported in Table 7.

In general the figures show that the stochastic volatility channel highly influences the nonlinear aspects of the model. But is it only the stochastic volatility that matters? Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012) analyze the accuracy of several solution methods in a neoclassical growth model with Epstein-Zin preferences and stochastic volatility. They report that higher-order approximations are needed to capture the non-linearities of the model. But in the long-run risk model, there are now two sources of non-linearities: the
Figure 3: Approximation Errors in the log Wealth-Consumption and Price-Dividend Ratio of the Bansal-Yaron Log-Linearization in the Model of Bansal, Kiku, and Yaron (2012a)

The graph shows isolines for the absolute errors in the log wealth-consumption ratio (left panel) and the log price dividend ratio (right panel) of the Bansal-Yaron log-linearization as a function of the states $x$ and $\sigma^2$ (black solid lines). The red shaded area shows the density of $10^7$ simulated observations of the states $x$ and $\sigma^2$ and the darker the red the higher is the concentration of the observations. The black pointed line marks the interval of all realized observations. Calibration of Bansal, Kiku, and Yaron (2012a).
Figure 4: Approximation Errors in the log Wealth-Consumption and Price-Dividend Ratio of the Bansal-Yaron Log-Linearization in the Model of Bansal and Yaron (2004)

The graph shows isolines for the absolute errors in the log wealth-consumption ratio (left panel) and the log price dividend ratio (right panel) of the Bansal-Yaron log-linearization as a function of the states $x$ and $\sigma^2$ (black solid lines). The red shaded area shows the density of $10^7$ simulated observations of the states $x$ and $\sigma^2$ and the darker the red the higher is the concentration of the observations. The black pointed line marks the interval of all realized observations. Calibration of Bansal and Yaron (2004).
Figure 5: Approximation Errors in the First Derivatives of the log Wealth-Consumption and Price-Dividend Ratio of the Bansal-Yaron Log-Linearization in the Model of Bansal, Kiku, and Yaron (2012a)

The graph shows isolines for the absolute errors in the first derivative of the log wealth-consumption ratio (left panel) and the log price-dividend ratio (right panel) with respect to the states $x$ and $\sigma^2$ of the Bansal-Yaron log-linearization (black solid lines). The red shaded area shows the density of $10^7$ simulated observations of the states $x$ and $\sigma^2$ and the darker the red the higher is the concentration of the observations. The black pointed line marks the interval of all realized observations. Calibration of Bansal, Kiku, and Yaron (2012a).
Figure 6: Approximation Errors in the First Derivatives of the log Wealth-Consumption and Price-Dividend Ratio of the Bansal-Yaron Log-Linearization in the Model of Bansal and Yaron (2004)

The graph shows isolines for the absolute errors in the first derivative of the log wealth-consumption ratio (left panel) and the log price-dividend ratio (right panel) with respect to the states $x$ and $\sigma^2$ of the Bansal-Yaron log-linearization (black solid lines). The red shaded area shows the density of $10^7$ simulated observations of the states $x$ and $\sigma^2$ and the darker the red the higher is the concentration of the observations. The black pointed line marks the interval of all realized observations. Calibration of Bansal and Yaron (2004).
Figure 7: Approximation Errors in the Second Derivatives of the log Wealth-Consumption and Price-Dividend Ratio in the Model of Bansal, Kiku, and Yaron (2012a)

The graph shows isolines for the absolute errors in the second derivative of the log wealth-consumption ratio (left panel) and the log price-dividend ratio (right panel) with respect to the states $x$ and $\sigma^2$ of the Bansal-Yaron log-linearization (black solid lines). The red shaded area shows the density of $10^7$ simulated observations of the states $x$ and $\sigma^2$ and the darker the red the higher is the concentration of the observations. The black pointed line marks the interval of all realized observations. Calibration of Bansal, Kiku, and Yaron (2012a).
The graph shows isolines for the absolute errors in the second derivative of the log wealth-consumption ratio (left panel) and the log price-dividend ratio (right panel) with respect to the states $x$ and $\sigma^2$ of the Bansal-Yaron log-linearization (black solid lines). The red shaded area shows the density of $10^7$ simulated observations of the states $x$ and $\sigma^2$ and the darker the red the higher is the concentration of the observations. The black pointed line marks the interval of all realized observations. Calibration of Bansal and Yaron (2004).
stochastic volatility channel and the long-run risk channel. Hence when solving the model, it is essential to understand which of the two components drives the non-linearities.

To answer this question we analyze the approximation errors implied by the Bansal-Yaron log-linearization for each of the two state variables of the model separately. In particular we first fix the stochastic volatility \( \sigma_t \) to its long-run mean, \( \bar{\sigma}^2 \), and second we solve the model without long-run risk, \( x_t = 0 \). Table 8 shows the corresponding errors in the unconditional mean and standard deviation of the log wealth-consumption and log price-dividend ratio for the two cases. We find that for the one-dimensional model with only long-run risks the approximation errors are very small with a maximum error of 0.21%. For the second case, without long-run risks and only stochastic volatility, the errors are slightly larger but still remain below 7.1%. Hence, compared to the maximum error of 26.9% for the full model, the errors are rather small. This finding suggests that neither the stochastic volatility alone nor the long-run risks component alone introduces the non-linearities in the model; instead it is the combination and interplay of the two features which makes the model so difficult to solve.

Table 8: Approximation Errors for Each State of the Long-Run Risks Model Separately

<table>
<thead>
<tr>
<th>State: ( x_t )</th>
<th>( E(w_t - c_t) )</th>
<th>( \sigma(w_t - c_t) )</th>
<th>( E(p_t - d_t) )</th>
<th>( \sigma(p_t - d_t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3.0e-5 )</td>
<td>( 2.4e-4 )</td>
<td>( 8.4e-4 )</td>
<td>( 0.0021 )</td>
<td></td>
</tr>
<tr>
<td>State: ( \sigma_t )</td>
<td>( 0.0014 )</td>
<td>( 0.0449 )</td>
<td>( 0.0262 )</td>
<td>( 0.0705 )</td>
</tr>
</tbody>
</table>

The table shows approximation errors in the unconditional mean and standard deviation of the log wealth-consumption and log price-dividend ratio induced by the Bansal-Yaron log-linearization in the long-run risks model for each of the two state variables \( x_t \) and \( \sigma_t \) separately. For the case with only \( x_t \) the parameters \( \sigma_t \) is simply set constant at its long-run mean \( \bar{\sigma}^2 \) (or equivalently \( v = \sigma_w = 0 \)). For the case with only \( \sigma_t \), \( x_t \) is set to 0 (or equivalently \( \rho = \phi_c = \Phi = 0 \)). All other parameter values are from the calibration of Bansal, Kiku, and Yaron (2012a).

4.4 Evaluation of Annualized Moments in the Long-Run Risks Model

In Table 9 we report the means and standard deviations of the annualized price-dividend ratio, the annualized market and risk-free return as well as the equity premium of the different solution methods for the updated calibration of the long-run risks model in Bansal, Kiku, and Yaron (2012a). The corresponding absolute relative errors of the moments are shown in Table 10. The log-linearized solution introduces a large approximation error and overstares the equity premium by more than 100 basis points and the volatility of the log price-dividend ratio by 6%. These values correspond to absolute relative errors of 22.5% (see Table 10). The discretization methods have even larger errors. For example, even for 50 nodes the Tauchen-Hussey method produces an error of 66.9% for the expected risk-free rate. The extension of
Floden (2007) leads to some smaller and some larger errors. Only the Tauchen method with 50 nodes has errors of similar size as the log-linearization; for example, it yields an error of 21.33% for the expected risk-free rate. It appears safe to say that the discretization with 30 or fewer nodes per state variable methods are ill-suited to solve the model. To obtain small errors, many more than 50 nodes per state variable are necessary. However, such an increase in the number of nodes will lead to very long running times and likely render the discretization methods impractical.

In stark contrast to the other methods, the errors for the projection methods are much smaller. For example, for the degree-6 approximation the largest error is 0.0043%, namely in the standard deviation of the price-dividend ratio for the Galerkin projection. Many of the reported errors are even one or more orders of magnitude smaller.

As the previous results have shown, the non-linearities of the long-run risk model are highly dependent on its parameters. The Bansal-Yaron log-linearization has been used for many variations of the model (Bansal, Kiku, and Yaron (2010), Bansal, Kiku, and Yaron (2012a), Bollerslev, Tauchen, and Zhou (2009), Kaltenbrunner and Lochstoer (2010), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), Bansal and Shaliastovich (2013), Bansal, Kiku, Shaliastovich, and Yaron (2014) and Beeler and Campbell (2012) among others). Also it is very difficult to accurately estimate long-run risk models so there is a lot of uncertainty about the true parameters of the model (Constantinides and Ghosh (2011), Grammig and Schaub (2014) and Bansal, Kiku, and Yaron (2012b)). Therefore, in Figures 9 and 10 we analyze the approximation errors implied by the Bansal-Yaron log-linearization with regard to changes in the parameters. In particular, we consider those parameters that are the main driving forces of the model, namely the risk aversion, $\gamma$, the intertemporal elasticity of substitution, $\psi$, and the serial correlation in the long-run risk, $\rho$, as well as the conditional variance, $\nu$. We find that for a risk aversion of approximately 5, the log-linearized solution basically coincides with the solution from the projection approach, which suggests that a linear solution gives a reasonable approximation to the model. However, when increasing the risk aversion the errors in the equity premium and the volatility of the log price-dividend ratio increase dramatically, with a large overestimate of both quantities. Furthermore, the accuracy depends highly on the persistence of the processes for both the long-run risk and the conditional volatility. We find that even very small changes can dramatically increase approximation errors. For example, in the original calibration with a persistence in the long run risk of $\rho = 0.975$ the overestimation of the equity premium is about 100 basis points (see Table 9). By slightly increasing $\rho$ to 0.98 however, the difference doubles with an overestimation of 200 basis points. For the persistence in the conditional variance, $\nu$, even a change of 0.0005, from 0.999 to 0.9995, increases the overestimation to 200 basis points. The figures also show that most of the results of the
Table 9: Annualized Moments in the Long-Run Risks Model

<table>
<thead>
<tr>
<th>n</th>
<th>$E(p_t - d_t)$</th>
<th>$\sigma(p_t - d_t)$</th>
<th>$E(r^m_t)$</th>
<th>$\sigma(r^m_t)$</th>
<th>$E(r_f^t)$</th>
<th>$\sigma(r_f^t)$</th>
<th>EP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collocation</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>3.25</td>
<td>0.24</td>
<td>5.78</td>
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<td>4.18</td>
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</table>

The table shows the mean and the standard deviation of the annualized log price-dividend ratio, the annualized market and risk-free return as well as the equity premium for the different solution methods and the parameter set as in Bansal, Kiku, and Yaron (2012a). All returns are shown in percent, so a value of 1.5 is a 1.5% annualized return.
Table 10: Absolute Relative Errors of the Annualized Moments in the Long-Run Risks Model

<table>
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<tr>
<th>n</th>
<th>$E(p_t - d_t)$</th>
<th>$\sigma(p_t - d_t)$</th>
<th>$E(r_m^t)$</th>
<th>$\sigma(r_m^t)$</th>
<th>$E(r_f^t)$</th>
<th>$\sigma(r_f^t)$</th>
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<td>6.2e-7</td>
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<td>3.4e-6</td>
<td>1.1e-6</td>
<td>1.6e-8</td>
</tr>
</tbody>
</table>

Collocation

| 3  | 3.0e-4         | 0.0182              | 0.0014     | 0.0020         | 0.0011     | 3.7e-4         | 0.0019|
| 6  | 1.3e-6         | 4.3e-5              | 1.0e-6     | 7.8e-7         | 2.4e-5     | 1.3e-5         | 6.7e-6|
| 9  | 1.1e-6         | 1.1e-6              | 2.6e-6     | 9.3e-9         | 9.6e-6     | 2.9e-6         | 9.4e-7|

Galerkin

| 3  | 0.0607         | 0.2250              | 0.1636     | 0.0129         | 0.1050     | 0.0148         | 0.2252|
| 6  | 0.2390         | 0.5068              | 0.3787     | 0.1863         | 0.7167     | 0.3179         | 0.6301|
| 50 | 0.1771         | 0.4584              | 0.3079     | 0.1725         | 0.6690     | 0.2573         | 0.5321|

BY Log-Linearization

| 10 | 0.4729         | 0.6118              | 0.5460     | 0.2206         | 0.8253     | 0.5614         | 0.8607|
| 30 | 0.2390         | 0.5068              | 0.3787     | 0.1863         | 0.7167     | 0.3179         | 0.6301|
| 50 | 0.1771         | 0.4584              | 0.3079     | 0.1725         | 0.6690     | 0.2573         | 0.5321|

Tauchen-Hussey

| 10 | 0.2116         | 0.4959              | 0.3468     | 0.2455         | 0.7809     | 0.3564         | 0.6057|
| 30 | 0.1757         | 1.3343              | 0.2538     | 0.1509         | 0.5624     | 0.0985         | 0.4411|
| 50 | 0.0789         | 0.5901              | 0.1315     | 0.1062         | 0.3480     | 0.0445         | 0.2416|

Tauchen-Hussey-Floden

| 10 | 0.1663         | 0.4901              | 0.2924     | 0.3328         | 0.8518     | 0.3183         | 0.5550|
| 30 | 0.0645         | 0.5495              | 0.1072     | 0.0768         | 0.2318     | 0.0214         | 0.1850|
| 50 | 0.0302         | 0.1410              | 0.0563     | 0.0773         | 0.2133     | 0.0013         | 0.1183|

Tauchen

The table shows absolute relative errors in the mean and standard deviation of the annualized price-dividend ratio, the annualized market and risk-free return as well as the equity premium for the different solution methods and the parameter set as in Bansal, Kiku, and Yaron (2012a).
long run risk model collapse for slight deviations from the model calibrations. For example, lowering the persistence, $\rho$, from 0.975 to 0.96 or $\nu$ from 0.999 to 0.99 reduces the equity premium from 4.74% to slightly above 2% and also the volatility in the log price dividend ratio decreases significantly for slightly lower persistence. Therefore it is especially important to solve the model accurately as small changes to the model can have large impacts on the higher-order dynamics of the model and hence introduce large approximation errors when using log-linear approximations.

In sum, using log-linearized approximations to solve long-run risk models can imply large errors, while projection methods provide highly accurate and robust solutions. Contrary to our experience with the endowment economy of Tallarini (2000), the large numerical errors of the log-linearization method have significant economic implications and substantially distort the relevant economic results. For example, the log-linearization approach overstates the equity premium by more than 100 basis points which is a relative error of more than 22%. Similarly, the log-linearization approach considerably overstates the volatility of the price-dividend ratio. The fact that the more accurate solution of the model leads to large deviations in the equilibrium values of the model would now require a recalibration of the model. That task is outside the focus of the current paper. Here we document that even very small changes in the model parameters can have large effects on equilibrium values; thus further applications of this class of models, require robust and accurate solution methods like the projection method presented in this paper.

5 Conclusion

This paper has investigated the significance of nonlinear dynamics for models with recursive preferences. Projection methods, a general tool for solving functional equations, are well-suited for the approximation of nonlinear pricing functions in asset-pricing models. We have found that the projection methods constructed in this paper outperform commonly used methods such as discretization and log-linearization in terms of efficiency and accuracy. These improvements become particularly significant for the latest generation of asset-pricing models, such as the influential long-run risks model. The increasing complexity of these asset-pricing models requires numerical solution methods, such as ours, that are robust to changes in model specification.

We have shown that the interplay of long run risks, stochastic volatility, and recursive preferences, three prominent model features in the recent asset-pricing literature, can result in rather nonlinear relationships between equilibrium quantities and the state variables. These nonlinearities make such models much more difficult to solve than previous generations of
Figure 9: Sensitivity of the Approximation Errors in the Long-Run Risks Model

The figure shows the equity premium obtained by the Bansal-Yaron log-linearization (dashed line) as well as the premium obtained by the collocation projection (solid line) as a function of the model parameters $\gamma, \psi, \rho$ and $\nu$, respectively, assuming that the other parameters are kept constant. The results are computed for the calibration of Bansal, Kiku, and Yaron (2012a) and in each panel, the black vertical line denotes the estimate used in original calibration.
The figure shows the volatility of the log price-dividend ratio obtained by the Bansal-Yaron log-linearization (dashed line) as well as the volatility obtained by the collocation projection (solid line) as a function of the model parameters $\gamma$, $\psi$, $\rho$ and $\nu$, respectively, assuming that the other parameters are kept constant. The results are computed for the calibration of Bansal, Kiku, and Yaron (2012a). The black vertical line in each panel denotes the estimate used in original calibration.
asset-pricing models in which equilibrium relationships are well approximated by linear functions.

Specifically, we have shown that while the Bansal-Yaron log-linearization provides a fast and easy solution method, its accuracy depends highly on the model specification. In the most recent calibration of the Bansal-Yaron long-run risks model (see Bansal, Kiku, and Yaron (2012a)), the approximation errors in the volatility of the log-price dividend ratio and the equity premium exceed 22%; these errors are economically significant. While discretization methods may, at least in theory, guarantee convergence of the approximate solution to the true equilibria, they are inefficient and highly dependent on the choice of parameters. They have particularly severe difficulties in the presence of highly persistent consumption processes and hence require a large number of discretization nodes. But since computation times increase dramatically with the number of nodes, particularly in higher dimensions, the discretization methods are all but impractical.

The solution methods presented in this paper prove to be highly accurate and the performance does depend neither on the choice of preference parameters nor on the specification of the underlying consumption processes. Already the degree-3 approximations yield highly accurate solutions in the asset-pricing models under consideration while they take only slightly longer than the log-linearization approach. The degree-6 approximations provide errors that are several magnitudes smaller than those of the other methods. In one dimension the difference between the Galerkin and the collocation projection is only marginal, but the Galerkin method proves to be more efficient for higher dimensions since it can be used together with complete polynomials instead of tensor products which are not easily implemented for collocation.

The results of this paper suggest that the solution methods that have been used in the past are not suitable for an accurate solution of modern asset-pricing models while the projection methods presented in this paper provide an efficient and robust alternative.

Appendix

A Projection Methods for Functional Equations

Projection methods (see Judd (1992) for an introduction or Chen, Cosimano, and Himonas (2014) for a brief overview) are a general tool to solve functional equations of the form

\[(Gz)(x) = 0, \] (20)
where the variable \( x \) resides in a (state) space \( X \subset \mathbb{R}^l, l \geq 1 \), and \( z \) is an unknown solution function with domain \( X \), so \( z : X \to \mathbb{R}^m \). The given operator \( G \) is a continuous mapping between two function spaces. Note that solving equation (20) requires finding an element \( z \) in a function space—that is, in an infinite-dimensional vector space.

The first central step of a projection method is to approximate the unknown function \( z \) on its domain \( X \) by a linear combination of basis functions. For the applications in this paper, it suffices to assume that the domain \( X \) is bounded and that the basis functions are polynomials.\(^7\) For a set \( \{\Lambda_k\}_{k \in \{0,1,\ldots,n\}} \) of chosen basis functions the approximation \( \hat{z} \) of \( z \) is

\[
\hat{z}(x; \alpha) = \sum_{k=0}^{n} \alpha_k \Lambda_k(x), \tag{21}
\]

where \( \alpha = [\alpha_0, \alpha_1, \ldots, \alpha_n] \) are unknown coefficients. Replacing the function \( z \) in equation (20) by its approximation \( \hat{z} \), we can define the residual function \( \hat{F}(x; \alpha) \) as the error in the original equation,

\[
\hat{F}(x; \alpha) = (G\hat{z})(x; \alpha). \tag{22}
\]

Instead of solving equation (20) for the unknown function \( z \), we now attempt to choose coefficients \( \alpha \) to make the residual \( \hat{F}(x; \alpha) \) zero. Note that instead of finding an element in an infinite-dimensional vector space we are now looking for a vector in \( \mathbb{R}^{n+1} \). Obviously, this approximation step greatly simplifies the mathematical problem.

This problem is unlikely to have an exact solution, so the second central step of a projection method is to impose certain conditions on the residual function, the so-called “projection” conditions, to make the problem solvable. In other words, the purpose of the projection conditions is to establish a set of requirements that the coefficients \( \alpha \) must satisfy. For a formulation of the projection conditions, define a “weight function” (term) \( w(x) \) and a set of “test” functions \( \{g_k(x)\}_{k=0}^{n} \). We can then define an inner product between the residual function \( \hat{F} \) and the test function \( g_k \),

\[
\int_X \hat{F}(x; \alpha)g_k(x)w(x)dx.
\]

This inner product induces a norm on the function space \( X \). Natural restrictions for the coefficient vector \( \alpha \) are now the projection conditions,

\[
\int_X \hat{F}(x; \alpha)g_k(x)w(x)dx = 0, \ k = 0, 1, \ldots, n. \tag{23}
\]

\(^7\)In addition to polynomial approximations, approximations using cubic splines or B-splines are often very useful.
Observe that this system of equations imposes \( n + 1 \) conditions on the \((n + 1)\)-dimensional vector \( \alpha \). Different projection methods vary in the choice of the weight function and the set of test functions. In this paper we use two different projections, the collocation and the Galerkin method.

A collocation method chooses \( n + 1 \) distinct nodes in the domain, \( \{x_k\}_{k=0}^n \), and defines the test functions \( g_k \) by

\[
g_k(x) = \begin{cases} 
0 & \text{if } x \neq x_k \\
1 & \text{if } x = x_k.
\end{cases}
\]

With a weight term \( w(x) \equiv 1 \), the projection conditions (23) simplify to

\[
\hat{F}(x_k; \alpha) = 0, \ k = 0, 1, \ldots, n. \tag{24}
\]

Simply put, the collocation method determines the coefficients in the approximation (21) by solving the square system (24) of nonlinear equations.

The Galerkin method uses the fact that Chebyshev polynomials are orthogonal on \([-1, 1]\) with respect to the inner product using the weight function \( w(x) \equiv \frac{1}{\sqrt{1-x^2}} \). Hence the Galerkin method uses the basis functions as the test functions, \( g_k(x) = \Lambda_k(x) \) and the projection conditions (23) become

\[
\int_X \hat{F}(x; \alpha) \Lambda_k(x) \frac{1}{\sqrt{1-x^2}} dx = 0, \ k = 0, 1, \ldots, n. \tag{25}
\]

## B Alternative Solution Methods

In this Appendix we provide a brief description of the alternative solution methods, namely the discretization methods by Tauchen (1986) and Tauchen and Hussey (1991) and the log-linearization approach as described in Bansal and Yaron (2004), considered in this paper. Similar to the projection algorithm, these three methods have to be conducted in two steps by first solving for the return on wealth, and then solving for the return of any individual asset.

### B.1 Discretization

The idea of discretization methods is to discretize the continuous state space by a finite number of discretization nodes and to design a Markov transition matrix for a Markov chain on the set of nodes. Put differently, these methods replace the continuous state space and conditional density functions by a discrete state space and transition probabilities, respectively. With the nodes and the Markov transition probabilities at hand, the pricing equation (11) becomes a square system of nonlinear equations. This nonlinear system has as many equations as nodes;
the unknown variables are the log price-dividend ratio at each node. We can then solve this system with a standard nonlinear equation solver.

Discretization methods differ in how they choose the discretization nodes and transition probabilities. For demonstration purposes, we consider the simple case of the discretization of an AR(1) process that is given by

$$x_{t+1} = (1 - \rho)\mu + \rho x_t + \epsilon_{t+1}, \quad \epsilon_t \sim N(0, \sigma_e^2),$$  \hspace{1cm} (26)

with persistence $|\rho| < 1$ and the unconditional mean $\mu$. The unconditional volatility of the process is given by $\sigma_x = \sigma_e / \sqrt{1 - \rho^2}$.

**Tauchen (1986)’s method**

Tauchen (1986) assumes a set of equally spaced nodes $X_{n_T} = \{x_1, \ldots, x_{n_T}\}$ for the discrete state space with $x_1 = \mu - m_T \sigma_y$ and $x_{n_T} = \mu + m_T \sigma_y$. The factor $m_T$ is a positive real number and determines the range of the state space. (To the best of our knowledge there is no optimal rule for choosing $m_T$ even though its value strongly influences the approximation results.) Denote the step size between two adjacent grid points by $h = x_i - x_{i-1}$. Then the elements $\pi_{ij}$ of the $(n_T \times n_T)$-transition probability matrix $\pi$ are defined by

$$\pi_{ij} = \begin{cases} 
\Phi \left( \frac{x_j + h/2 - (1 - \rho)\mu - \rho x_i}{\sigma_e} \right) 
& \text{for } j = 1, \\
\Phi \left( \frac{x_j + h/2 - (1 - \rho)\mu - \rho x_i}{\sigma_e} \right) - \Phi \left( \frac{x_j - h/2 - (1 - \rho)\mu - \rho x_i}{\sigma_e} \right) 
& \text{for } 1 < j < n_T, \\
1 - \Phi \left( \frac{x_j - h/2 - (1 - \rho)\mu - \rho x_i}{\sigma_e} \right) 
& \text{for } j = n_T.
\end{cases}$$

**Tauchen and Hussey (1991)’s method and the extension by Floden (2007)**

Tauchen and Hussey (1991)’s method is based on Gauss-Hermite quadrature. Let $\xi_i$ and $\omega_i$, $i = 1, \ldots, n_{TH}$, be the Gauss-Hermite nodes and weights on the interval $[-\infty, +\infty]$, respectively. The approximation nodes are then given by $x_i = \mu + \sqrt{2}\sigma_e \xi_i$ and the entries $\pi_{ij}$ of the transition probability matrix $\pi$ can be computed by

$$\pi_{ij} = \frac{\hat{\omega}_{ij}}{\sum_{j=1}^N \hat{\omega}_{ij}}$$  \hspace{1cm} (27)

with

$$\hat{\omega}_{ij} = \pi^{-0.5} \omega_j \frac{f(x_j|x_i)}{f(x_j|\mu)}$$  \hspace{1cm} (28)

where $f(\cdot|x_i)$ is the density function of $N((1 - \rho)\mu + \rho x_i, \sigma^2)$. Tauchen and Hussey (1991) simply choose $\sigma = \sigma_e$. Floden (2007) chooses $\sigma = a\sigma_e + (1 - a)\sigma_x$ with $a = 0.5 + 0.25\rho$,
so $\sigma$ is a weighted average of $\sigma_x$ and $\sigma_e$. He claims that his approach performs significantly better than the original Tauchen and Hussey (1991) method, particularly for highly persistent processes.

### B.2 Log-Linearization as in Bansal and Yaron (2004)

As described in Section 2.3 constructing linear approximations to models with recursive preferences and stochastic volatility is not straight forward. Bansal and Yaron (2004) present a solution method that linearizes around the stochastic steady state of the model and provides policies that are dependent on the volatility of the underlying process. In the following we briefly describe their approach. For a detailed description please refer the appendix of the original paper.

Assume that the log price-dividend ratio $z_i(x)$ of asset $i$ is a linear function of the state variable $x \in \mathbb{R}^l$,

$$z_i(x) = A_{0,i} + A_{1,i}x \quad (29)$$

where $A_{0,i}$ is an unknown constant, $x$ is an $l \times 1$ vector representing the state of the economy in period $t$ and $A_{1,i}$ is a $1 \times l$ vector of unknown slope coefficients. Define the current state by $x$ and the state in the subsequent periods by $x'$. The return of asset $i$ is then given by (see equation (10)),

$$r_i(x'|x) = \log \left( e^{z_i(x')} + 1 \right) - z_i(x) + \Delta d_i(x'|x). \quad (30)$$

Bansal and Yaron (2004) use the Campbell and Shiller (1988) approximation to linearize the return equation:

$$r_i(x'|x) \approx \kappa_{i,0} + \kappa_{i,1} z_i(x') - z_i(x) + \Delta d_i(x'|x), \quad (31)$$

where the constants $\kappa_{i,0}$ and $\kappa_{i,1}$ depend only on the average log price-dividend ratio $\bar{z}_i$:

$$\kappa_{i,1} = \frac{e^{\bar{z}_i}}{1 + e^{\bar{z}_i}} \quad (32)$$

$$\kappa_{i,0} = - \log \left( (1 - \kappa_{i,1})^{1-\kappa_{i,1}} \kappa_{i,1} \right). \quad (33)$$

After substituting expressions (29) and (31) into the pricing equation (11), the unknown coefficients $A_{0,i}$ and $A_{1,i}$ can be derived analytically as a function of the linearization parameters $\kappa_{i,0}$ and $\kappa_{i,1}$ by collecting all the terms for each state variable and using the fact that the Euler equation has to hold in each state of the economy. Taking expectations of equation (29) we have that the parameters $A_{0,i}$ and $A_{1,i}$ determine $\bar{z}$ while $\kappa_{i,0}$ and $\kappa_{i,1}$ are nonlinear functions of $\bar{z}$. Hence, by iteratively computing a fixed point we can solve for $A_{0,i}$, $A_{1,i}$, $\kappa_{i,0}$, $\kappa_{i,1}$ and $\bar{z}$ simultaneously. For the return of the wealth portfolio the approximation can be applied.
analogously. As the return is given by

\[ r_w(x'|x) = z_i(x') - \log \left( e^{z_i(x') - 1} \right) + \Delta c(x'|x). \]  

(34)

the approximation becomes

\[ r_w(x'|x) \approx z_i(x') + \kappa_{i,0} - \kappa_{i,1} z_i(x) + \Delta d_i(x'|x), \]  

(35)

with

\[ \kappa_{w,1} = \frac{e^{\tilde{z}_w}}{e^{	ilde{z}_w} - 1}, \]  

(36)

\[ \kappa_{w,0} = \log \left( (\kappa_{w,1} - 1)^{1-\kappa_{w,1}} \kappa_{w,1}^{\kappa_{w,1}} \right). \]  

(37)

C Coefficients for the Bansal-Yaron Log-Linearization

C.1 Log-Linearization Coefficients for the Endowment Economy of Tallarini (2000)

Parameters of the BY Log-Linearization as derived in Appendix B for the Endowment Economy of Tallarini (2000) considered in Section 3:

\[ A_{0,w} = \frac{\log \delta + (1 - \frac{1}{\psi})\mu + 0.5\theta((1 - \frac{1}{\psi}) + A_1)^2\sigma^2 + \kappa_{w,0}}{\kappa_{w,1} - \rho} \]  

(38)

\[ A_{1,w} = \frac{(1 - \frac{1}{\psi})(\rho - 1)}{1 - \kappa_{w,1}\rho} \]  

(39)
C.2 Log-Linearization Coefficients for the Long-Run Risks Model

Parameters of the BY Log-Linearization as derived in Appendix B for the long-run risks model of Bansal and Yaron (2004) considered in Section 4:

\[ A_{0,w} = \frac{\log \delta + (1 - \frac{1}{\psi})\mu + A_{2,w}\bar{\sigma}^2(1 - \nu) + \kappa_{0,w} + 0.5\theta(A_{2,w}\sigma_w)^2}{\kappa_{1,w} - 1} \]

\[ A_{1,w} = \frac{(1 - \frac{1}{\psi})}{\kappa_{1,w} - \rho} \]

\[ A_{2,w} = \frac{0.5\theta ((1 - \frac{1}{\psi})^2 + (A_{1,w}\phi_e)^2)}{\kappa_{1,w} - \nu} \]

\[ A_{0,m} = \left[ \theta \log \delta - \gamma\mu + \mu_d + (\theta - 1)(\kappa_{0,w} + A_{0,w}(1 - \kappa_{1,w})) + (\theta - 1)A_{2,w}\sigma^2(1 - \nu) + \kappa_{0,m} + \kappa_{1,m}A_{2,m}\bar{\sigma}^2(1 - \nu) + 0.5 ((\theta - 1)A_{2,w}\sigma_w + \kappa_{1,m}A_{2,m}\sigma_w)^2 \right] / (1 - \kappa_{1,m}) \]

\[ A_{1,m} = \frac{(\Phi - \frac{1}{\psi})}{1 - \kappa_{1,m}\rho} \]

\[ A_{2,m} = \frac{0.5((\pi - \gamma)^2 + \phi^2) + 0.5((\theta - 1)A_{1,w}\phi_e + \kappa_{1,m}A_{1,m}\phi_e)^2 + (\theta - 1)A_{2,w}(\nu - \kappa_{1,w})}{1 - \kappa_{1,m}\nu} \]

\[ A_{0,rf} = \theta \log \delta - \gamma\mu + (\theta - 1)(\kappa_{0,w} + A_{0,w}(1 - \kappa_{1,w}) + A_{2,w}\bar{\sigma}^2(1 - \nu)) + 0.5((\theta - 1)A_{2,w}\sigma_w)^2 \]

\[ A_{1,rf} = -\gamma + (\theta - 1)A_{1,w}(\rho - \kappa_{1,w}) \]

\[ A_{2,rf} = 0.5(\gamma^2 + ((\theta - 1)A_{1,w}\phi_e)^2) + (\theta - 1)A_{2,w}(\nu - \kappa_{1,w}) \]
References


